4. In two recent papers (L. M. Blumenthal: *A Complete Characterization of Proper Pseudo D-Cyclic Sets of Points*, American Journal of Mathematics, vol. 54 (1932), pp. 387–396; and *Concerning Regular Pseudo D-Cyclic Sets*, American Journal of Mathematics, vol. 54, pp. 729–738) theorems characterizing d-cyclic and pseudo d-cyclic sets of points were proved by synthetic methods. In this paper the characterization is obtained by means of determinants. Apart from interpreting analytically the theorems of the papers referred to, the methods used here have the advantage of being readily extended to effect a characterization of the n-dimensional spherical surface S_n. We designate by $\Delta(1, 2, \cdots n)$ the following determinant:

$$\begin{vmatrix} 1 & \cos\frac{p_1p_2}{r} & \cos\frac{p_1p_3}{r} \cdots \cos\frac{p_1p_n}{r} \\ \cos\frac{p_2p_1}{r} & 1 & \cos\frac{p_2p_3}{r} \cdots \cos\frac{p_2p_n}{r} \\ \cos\frac{p_3p_1}{r} & \cos\frac{p_3p_2}{r} & 1 \quad \cdots \cos\frac{p_3p_n}{r} \\ \cdots & \cdots & \cdots \\ \cos\frac{p_np_1}{r} & \cos\frac{p_np_2}{r} & \cos\frac{p_np_3}{r} \cdots 1 \end{vmatrix}$$

where the distance $p_ip_j (i, j = 1, 2, \cdots n)$ is measured as the length of the shorter arc joining p_i and p_j. The following theorems are proved:

Theorem 1: Three points p_1, p_2, p_3 are congruent to three points of a circle of radius r if and only if $p_ip_j/r \leqq \pi (i, j = 1, 2, 3)$ and $\Delta(1, 2, 3)$ equals zero.

Theorem 2: Four points p_1, p_2, p_3, p_4 are congruent to four points of a circle of radius r if and only if each three points is congruent to three points of the circle and $\Delta(1, 2, 3, 4)$ equals zero.

Theorem 3: If all four third-order principal minors of the determinant $\Delta(1, 2, 3, 4)$ are equal to zero, and $p_ip_j/r \leqq \pi$, then $\Delta(1, 2, 3, 4)$ either vanishes or has the value

$$-4 \sin^2\frac{p_1p_2}{r} \sin^2\frac{p_1p_3}{r} \sin^2\frac{p_2p_3}{r},$$

in which case $p_ip_j/r \neq \pi$ and

$$\cos^2\frac{p_1p_2}{r} = \cos^2\frac{p_3p_4}{r}, \quad \cos^2\frac{p_1p_3}{r} = \cos^2\frac{p_2p_4}{r}, \quad \cos^2\frac{p_2p_3}{r} = \cos^2\frac{p_1p_4}{r}.$$

It is shown that the three kinds of pseudo d-cyclic quadruples may be obtained by reflections in a circle.

5. In this paper Professor Reinsch showed that the fundamental psychological principles and laws of thought, memory, habit, and development of

abilities govern in the methods of teaching. By means of specific illustrations in geometry, algebra, trigonometry, and calculus, he demonstrated how these psychological principles are to be employed in achieving some of the important aims in teaching mathematics as: effective acquisition of knowledge; thorough understanding of principles; ability to use mathematics as a tool; ability to think carefully and correctly; ability to do independent thinking; development of judgment; appreciation of beauties in mathematics; development of desirable habits of alert attention, questioning attitude, self-activity.

6. In view of the proposed change in secondary mathematics in Texas, a study of the efficiency of the present system might serve as a basis for reorganization. A study of failures in freshman mathematics for the State of Texas during the years 1925–26, 1927–28, 1928–29, and 1929–30 (See Committee of Deans' Report—Proceedings of the Secondary Schools and Colleges in the Southern States) shows that 26.2 per cent of those students who went directly from high school to college were not able to receive a passing grade in their first attempt. In 1929–30, 41.6 per cent either failed or received a barely passing grade, both of which were unsatisfactory.

It is the consensus of opinion among college teachers of mathematics (See J. Seidlin, Mathematics Teacher, Dec. 1932) and science that the secondary schools produce graduates with the following general characteristics:

(1) Worn out or weary of mathematics,
(2) No inspiration for individual investigation,
(3) No appreciation of accuracy,
(4) Not able to place a decimal point in its proper place,
(5) Direct and inverse proportions are meaningless.

The following suggestions might make secondary mathematics more effective:

(1) Reorganization of Junior and Senior High School mathematics so that it will be useful and inspirational, and at the same time contain the fundamental notions for scientific study.

(2) College teachers should recommend only those students for teaching positions in secondary mathematics who have had two, and preferably three years of college mathematics.

NAT EDMONSON, *Secretary*

THE EIGHTEENTH ANNUAL MEETING OF THE OHIO SECTION

The eighteenth annual meeting of the Ohio Section of the Mathematical Association of America was held at the Ohio State University, Columbus, Ohio, Thursday, April 6, 1933. An afternoon session, dinner, and evening session were held, with the chairman of the Section, Professor O. L. Dustheimer, presiding.

The Section had as its guests Professor and Mrs. H. T. Davis of the University of Indiana.

In spite of the prevailing unfavorable economic conditions, the attendance at this meeting, so far as members of the Association were concerned, was the largest in the history of the Section, while the total registration was the largest since 1920. Eighty-six persons were registered as being in attendance, including the following fifty-seven members of the Association: R. B. Allen, W. E. Anderson, F. R. Bamforth, Grace M. Bareis, I. A. Barnett, H. M. Beatty, L. T. Black, Henry Blumberg, M. G. Boyce, J. B. Brandeberry, R. S. Burington, O. E. Brown, F. E. Carr, T. F. Cope, F. F. Crandell, Rufus Crane, Wayne Dancer, H. T. Davis, O. L. Dustheimer, P. S. Dwyer, F. J. Feinler, C. W. Foard, T. M. Focke, B. C. Glover, Harris Hancock, R. C. Hildner, E. J. Hirschler, F. C. Jonah, Margaret E. Jones, E. M. Justin, L. C. Knight, H. W. Kuhn, A. C. Ladner, Lincoln LaPaz, R. H. MacCullough, C. C. MacDuffee, R. E. Manchester, Florentina Mathias, C. N. Moore, C. C. Morris, Max Morris, J. R. Musselman, R. L. Newlin, Jesse Pierce, Tibor Radó, S. E. Rasor, Y. K. Roots, S. A. Rowland, W. G. Simon, Mary E. Sinclair, Ruth B. Smyth, H. E. Stelson, C. F. Thomas, C. L. Weaver, J. H. Weaver, R. B. Wildermuth, F. B. Wiley.

The following officers were elected for the coming year: Chairman, I. A. Barnett, University of Cincinnati; Secretary-Treasurer, Rufus Crane, Ohio Wesleyan University; Member of Executive Committee, F. R. Bamforth, Ohio State University; Member of Program Committee, J. R. Musselman, Western Reserve University.

It is expected that the next meeting of the Section will be held at the Ohio State University on Thursday, April 5, 1934.

The afternoon session was given to the reading of papers. The evening session was given to the discussion of the report of a committee that has been working for three years on a study of the mathematical preparation of students entering our colleges from the high schools of Ohio.

The following papers were read at the afternoon session:

1. "Applied mathematics in a liberal arts college" by the Chairman of the Section, Professor O. L. Dustheimer, Baldwin-Wallace College.

2. "The life and work of Eliakim Hastings Moore" by Professor W. G. Simon, Western Reserve University.

3. "Undergraduate preparation for graduate work in analysis" by Professor C. N. Moore, University of Cincinnati.

4 "The expanding universe" read by title on account of the absence of the author, Professor J. J. Nassau, Case School of Applied Science.

5. "The mathematical representation of a score in statistics" by Professor H. A. Toops, Ohio State University.

6. "Approximate methods for finding simple interest on installments" by Professor H. E. Stelson, Kent State College.

7. "The sum of the mth powers of the first n natural numbers" by C. L. Weaver, Kent State College.

8. "Some applications of complex numbers to real geometry of the plane" by M. G. Boyce, Western Reserve University.

9. "The predictable element in economic series" by Professor H. T. Davis, Indiana University, by invitation of the Program Committee.

Abstracts of these papers follow:

1. Since mathematics is no longer compulsory in our schools, teachers of mathematics must "sell" mathematics to the student. They must make their classes so interesting and beneficial that students will elect mathematics.

Too many of our educators have no conception of the role which mathematics has played in the development of our present civilization and its possibilities for future applications. Chemistry and biology are rapidly becoming exact sciences. All chemical phenomena are now being interpreted in terms of physical chemistry, which can be mastered only when the calculus is applied. In biology differential equations are used in the study of metabolic activities, digestion, growth, genetic equilibrium and surface tensions. Mathematical methods are rapidly being applied in education, psychology, psychophysics, economics, and even sociology.

The modern teacher of mathematics has a real opportunity in the training of students in the use of mathematical tools that will help him in so many other departments. We must constantly recall that our major task is to educate our students in the concepts, methods, and applications of the "queen of the sciences."

2. In his paper Professor Simon gave a personal appreciation of Professor E. H. Moore, based on impressions gained from Moore's reputation as a student in Woodward High School in Cincinnati, and also from his own contact with Moore while a student of his in Chicago.

3. In this paper Professor Moore stressed the importance of avoiding an excess of formalism in such courses as introductory and advanced calculus. He urged that the spirit of analysis be preserved, even in elementary courses in the subject, and that so-called proofs based on geometric intuition be avoided. For students who concentrate on analysis in their undergraduate work it would seem desirable, and in many cases feasible, to introduce into their work some of the more recent developments of analysis, such as the Lebesgue integral and the use of divergent series.

4. One may summarize our knowledge of the expanding universe by answering the following questions: What is it that expands? What are the observational evidences of the expansion? What is the rate of expansion? What are the theories regarding it? The last of these questions will be considered here.

In 1917, Professor Einstein modified the law of gravitation, $G_{\mu\nu}=0$ to $G_{\mu\nu}=\lambda g_{\mu\nu}$ in order to avoid the difficulties of the boundary solution at infinity. This new equation makes the world static, finite, and saturated with mass.

Another solution of the same equation was put forward in the same year by Professor de Sitter which makes the world expanding but without mass. In 1923, Professor Eddington and Professor Weyl established independently the fact that the λ was necessary and that it is a factor that makes measurements of length relative. Assuming a non-static world, Professor LeMaitre brought forward in 1928 a solution of the same equation which makes the world expanding but this time with a mass of uniform distribution.

In 1931, Eddington bridged this with the quantum theory, and by utilizing the idea that distances within atoms must be functions of λ to produce relative measures he established the equation $\sqrt{N}/R = mc^2/e^2$ where N is the number of electrons in the universe, R is the initial radius of the universe, c is the velocity of light, and m and e are the mass and charge of an electron. That is, from laboratory values only, he is able to determine a relation between N and R, from which the value of the rate of recession could be established.

A joint paper by Einstein and de Sitter in 1932 brings out the fact that the factor λ may be equal to zero and we may still be able to explain the recession of the galaxies. The latest contribution (1933) on the theoretical side of this problem is the work of Professor Tolman who introduces a pulsating universe.

The story of the Greek letter λ is a fascinating one, and is admirably told in Eddington's new book, "The Expanding Universe."

5. A series of statistical classes represented by their face-values, X_F, may be represented for computational purposes by a series of coded scores, X', defined by the relationship, $X_F = IX' + F_0'$, where I is signed, (positive if the X' series ascends with ascending values of X_F, and vice versa), the width of each class; F_0' is the face-value of the class corresponding to the origin of the X' series. Since only 40 such variables may be punched into a Hollerith card, some 240 variables may be punched into a Hollerith card by employing multiple geometrical punching, where any one such X' score is defined as

$$X_{F_1}' = a_1 2^0 + b_1 2^1 + c_1 2^2 + d_1 2^3,$$

where the coefficients, a_1, b_1, etc., are restricted to zero and one (zero for no punch, one for punch). All possible moments may be computed from the formulae resulting from a manipulation of the above basic relationship.

6. The practice of using simple interest over short time intervals justifies the use of approximation formulae for computing simple interest in short time installment purchases. This may be as much as ten to fifteen percent below the compound interest in actual cases. Three approximation formulae can be used to give fairly satisfactory results. The value of i as given by the equation

$$K = \sum_{x=1}^{n/r} (1 + xri)^{-1}$$

can be approximated by Newton's and the Euler-Maclaurin formulae. The simple interest rate can be computed from the discount rate by use of average time.

7. An expansion of x^m may be secured by Newton's advancing difference formula. The finite integral of this expression may be used to sum the mth powers of the first n natural numbers.

$$\sum_1^n x^m = \Delta^{-1}x^m \Big]_1^{n+1} = \Delta 0^m(n+1)_2 + \Delta^2 0^m(n+1)_3 + \Delta^3 0^m(n+1)_4 + \cdots$$

The terms of this expansion are symmetrical and the coefficients are easily determined. This expression is somewhat easier to use than the expansions usually found.

8. The principal object of this paper was a classification of triads of points by means of invariants of the cubics whose roots are co-ordinates of the points in the Argand diagram. The cases treated included those of coincident and collinear points, and points that are the vertices of equilateral, isosceles, and right triangles. A brief discussion was also given of certain parametric equations of central conics.

9. In an address illustrated by lantern slides, Professor Davis showed how the periodic structure of economic series (for example, stock market averages, pig iron production indices, price averages, etc.) can be studied by means of periodogram analysis, difference equations, and the harmonic analysis of lag-correlations. It was shown that high correlations between such series as the stock market average and the pig iron production index are due in large part to coincident periods in the two series. The analysis of lag-correlations reveal these coincident periods, while the analysis of systems of difference equations reveal the non-coincident periods as elastic interactions. In discussing the dangers of too confident prediction in economic matters, Professor Davis showed that a random series smoothed by a twelve-months' moving average had a period almost exactly equal to the fundamental period of the Bradstreet commodity price index. He also touched on the significance of straightline and logistic trend lines and showed how the present decline in the price index was a phenomenon probably predictable on the basis of long-time trend lines. The computations involved in the paper were furnished by the laboratory of the Cowles Commission for Research in Economics.

RUFUS CRANE, *Secretary*

THE APRIL MEETING OF THE ROCKY MOUNTAIN SECTION

The seventeenth annual meeting of the Rocky Mountain Section was held at Colorado Agricultural College, Fort Collins, Colorado, on Friday and Saturday, April 14–15, 1933. Professor A. G. Clark presided at each of the three sessions.

The attendance was thirty-five, including the following twenty-one members of the Association: C. F. Barr, Jack Britton, A. G. Clark, I. M. DeLong,

J. C. Fitterer, G. W. Gorrell, I. L. Hebel, C. A. Hutchinson, Louise Johnson, A. J. Kempner, Claribel Kendall, A. J. Lewis, S. L. Macdonald, A. S. McMaster, W. K. Nelson, Greta Neubauer, E. D. Rainville, O. H. Rechard, A. W. Recht, Mary S. Sabin, C. H. Sisam.

Members and friends of the Association were guests of the College on the evening of April 14. At the business session, Professor C. H. Sisam of Colorado College, was elected Chairman for the coming year, and Professor J. C. Fitterer, of Colorado School of Mines, was elected Vice-Chairman.

The following ten papers were read:

1. "A solution of a system of linear matrix equations in two unknowns," by Miss Rachel Achenbach, University of Wyoming, by invitation.
2. "On foci of algebraic curves with applications to cubic curves" (Thesis presented by Ethel A. Rice for M. A. University of Colorado) by Professor Claribel Kendall, University of Colorado.
3. "Entropy, strain and the Pauli exclusion principle" by Professor Guy Berry, Colorado Agricultural College, by invitation.
4. "Notes on Riccati's differential equations" by E. D. Rainville, University of Colorado.
5. "A theorem on point-wise discontinuous functions" by Professor O. H. Rechard. University of Wyoming.
6. "On Graeffe's method of solution of algebraic equations" by Professor C. A. Hutchinson, University of Colorado.
7. "The use of calculators in solving Kepler's problem" by Professor A. W. Recht, University of Denver.
8. "The 'Zig' function of Wirth" by Professor C. F. Barr, University of Wyoming.
9. "The solution of algebraic equations by infinite series" by Professor A. J. Lewis, University of Denver.
10. "Symmetric functions and resultants" by Professor C. H. Sisam, Colorado College.

Abstracts of the papers follow below, the numbers corresponding to the numbers in the list of titles:

1. The purpose of this paper is to develop methods of solution for the sixteen systems of linear matrix equations in two unknowns which result from the corresponding ordinary algebraic system when the constants and unknowns in these latter equations are replaced by matrices of the nth order. Two of the systems yielded to solution by the methods of elimination by addition and by substitution. The other fourteen systems were reduced to a single standard form which was solved by the use of the Hamilton-Cayley equation.

2. In this paper, Miss Rice was particularly interested in the graphical representation of the location of the real foci of certain cubic curves. For practical purposes in the matter of computation the cubic curves chosen were sym-

metric with respect to a point or an axis. The foci of some twenty-four such cubics of varying types were given. The discussion of these special cubics was preceded by a summary of the known results concerning the number of foci of algebraic curves in general.

3. The author shows, by using the geometrical weight method developed by Kimball for an ideal gas, that the entropy of a real gas is proportional to the strain, and that the equations for maximum entropy are equilibrium equations between stress and strain. The velocity distribution function is the same as that of the Fermi-Dirac statistics. This method offers an explanation of the second law of thermodynamics and the Pauli exclusion principle.

4. In the general Riccati Equation $dy/dx = A_0(x) + A_1(x)y + y^2$, it is most desirable to obtain a simple particular solution. If this is available, the complete solution follows almost at once. When A_0 and A_1 are polynomials, it is reasonable to search for polynomial solutions. In these notes Mr. Rainville shows that never more than one or two polynomials, for which simple formulas are given, need be tested as trial solutions. Some extensions varying the functional form of A_0 and A_1 are treated.

5. In this paper there is presented a statement and proof of the following theorem: Given a function, $f(x)$, continuous over a residual point set on an interval (a, b); if the function is defined at the remaining points of the interval by the "closest approximating function" method, it will be point-wise discontinuous on the interval.

6. This is the paper read by title at the Los Angeles Meeting of the Association. An abstract appeared in this MONTHLY, November 1932, p. 503; and the complete paper will appear in a later issue.

7. A demonstration showing how J. Peter's 7-place table of natural trigonometric functions, in which the argument is given in decimals of a degree, combined with the modern calculator, greatly facilitate the solution of Kepler's well-known equation $M = E - e \sin E$.

8. This paper presents the function

$$\operatorname{zig} u = 2\{[R_{u/2}] + (-1)^{[R_{u/2}]}R_{u/2}\}$$

and its derivative; in which $u = f(x)$, $[s]$ means the greatest integer in s, and R_u means $u - [u]$. Its adaptibility to special configurations was demonstrated by using it in the polar equation

$$\rho = a \cos\frac{\pi}{n} \Big/ \cos\left\{\frac{\pi}{n}\left(1 - \operatorname{zig}\frac{n\theta}{\pi}\right)\right\}$$

to map a regular n-sided polygon in a circle of radius a. The function zig u was developed and named by Mr. Don Wirth.

9. This paper outlines methods of expressing all the roots of an algebraic equation by infinite series and formulates conditions of convergence for these series.

10. This paper deals with a proof of the possibility of representing a given integral symmetric function in terms of the elementary symmetric functions and with a method of determining this representation by the use of resultants.

A. J. LEWIS, *Secretary*

THE TWENTY-SECOND MEETING OF THE IOWA SECTION

The twenty-second meeting of the Iowa Section of the Mathematical Association of America was held with the Iowa Academy of Science at Coe College, Cedar Rapids, Iowa, on Friday and Saturday, April 21 and 22, 1933. The meetings were held in room 117, Science Hall.

The attendance was about fifty, including the following twenty-one members of the Association: R. P. Baker, E. W. Chittenden, L. M. Coffin, N. B. Conkwright, C. W. Emmons, Cornelius Gouwens, M. E. Graber, I. J. Gwinn, Gertrude A. Herr, Dora E. Kearney, O. C. Kreider, F. M. McGaw, Arthur Ollivier, J. F. Reilly, H. L. Rietz, W. J. Rusk, E. R. Smith, C. W. Strom, John Theobald, L. E. Ward, Roscoe Woods.

The Section Chairman, Professor L. M. Coffin, presided at both the Friday afternoon and Saturday morning sessions. Dinner was enjoyed together Friday evening in the Jefferson Room, Hotel Roosevelt. The officers elected for 1933–1934 are as follows: Chairman, J. F. Reilly, University of Iowa; Vice-Chairman, M. E. Graber, Morningside College; Secretary-Treasurer, Cornelius Gouwens, Iowa State College.

A committee consisting of Professors R. P. Baker and Roscoe Woods prepared the following statement relative to the death of Daniel Kreth: "Daniel Kreth, engineer and surveyor, of Wellman, Iowa, a charter member of the Association, died in 1932. From 1914 to 1924 Mr. Kreth was an active contributor to the Monthly of problems and solutions. His interest in mathematics showed itself not only by his activities in the Association but also in the collection of a library. The Iowa Section laments his passing as a member and feels keenly the loss of inspiration which comes from knowing a man who derived a great deal of pleasure from his study of mathematics."

The program consisted of fourteen papers, as follows:

1. "On the resolution of $4X = Y^2 - (-1)^{(p-1)/2}\, pZ^2$ for $p = 67$, 71, and $X = (x^p - 1)/(x - 1)$" by Professor Cornelius Gouwens, Iowa State College.

2. "Some properties of the logarithmic potential" by Professor J. J. Westemeier, Des Moines Catholic College, by invitation.

3. "The teaching of the trigonometric functions of $2x$ and of $x/2$" by Professor Roscoe Woods, University of Iowa.

4. "Sophus Lie's geometry of imaginaries" by Professor M. E. Graber, Morningside College.

5. "A problem in simple harmonic motion of a particle moving in a medium of varying density" by Robert MacAllister, Wartburg College, by invitation.

6. "Runge's method" by Professor N. B. Conkwright, University of Iowa.

7. "A system of circles inscribed in an annulus" by Professor C. W. Emmons, Simpson College.

8. "An Iowa journal of mathematics" by Professor Gertrude A. Herr, Iowa State College.

9. "Zeros of the Hermitian polynomials" by Professor E. R. Smith, Iowa State College.

10. "An approach to a class in freshman mathematics" by Father John Theobald, Columbia College.

11. "A test for significance in a unique sample" by Professor A. E. Brandt, Iowa State College, by invitation.

12. "Devices in mathematical instruction" by Professor F. M. McGaw, Cornell College.

13. "The nature of probability" by Professor E. S. Allen, Iowa State College, by invitation.

14. "Note on the purchase price of a bond" by Professor J. F. Reilly, University of Iowa.

Abstracts of some of the papers follow, the numbers corresponding to the numbers in the list of titles:

1. In 1930 Professor Gouwens reported on the cases for $p=41$ and $p=43$ but later learned that those had been given by Pocklington in Nature, vol. 107, (1921). In this paper the resolutions for $p=67$ and $p=71$ are presented.

2. In this paper Professor Westemeier presented the elegant elementary methods developed by Erhard Schmidt (*Bemerkung Zur Potentialtheorie*, H. A. Schwarz Gewidmet, 1914, p. 364–383) to obtain the properties near the acting masses of the integrals defining the logarithmic potentials of simple plane curve, double plane curve, and plane area distributions.

6. Professor Conkwright gave an exposition suitable for class room use of Runge's method of numerical approximation to the solution of the differential equation $y'=f(x, y)$.

7. In his paper Professor Emmons developed the relations among R, R', r, n, and α, where R and R' are radii of the inner and outer bounds of an annulus, r, the radius of a circle inscribed in the annulus, n, the number of circles tangent consecutively to each other and to the annulus, and 2α, the angle at the center of the annulus subtended by the circles. Special consideration was given to integral and non-integral values of n.

8. The Analyst, A Monthly Journal of Pure and Applied Mathematics, was published in Des Moines from January 1874 to November 1883 by Dr. Joel E. Hendricks, a self taught mathematician. From the correspondence of Dr. Hendricks, from his scrap-book, and from interviews with his daughters and grandchildren, Miss Herr has collected some interesting facts concerning the author and the publication. The character of the journal is discussed from the standpoint of the contributions and the types of articles.

9. If one substitutes $e^{x^2/2}\mu_n(x)$ for $P_n(x)$ in the differential equation $P_n''(x) - 2xP_n'(x) + 2nP_n(x) = 0$, which the Hermitian polynomials satisfy, there is obtained a new equation $\mu_n''(x) + (1 + 2n - x^2)\mu_n(x) = 0$. Professor Smith showed how the values of μ could be approximately determined by extrapolation based on corresponding values for polynomials of lower order, and how the values of the roots thus obtained could be corrected by the Newton-Raphson method. Professor Smith presented a table of roots computed to five decimal places.

10. Experience shows that it is unwise to expect much mathematical background in the case of the average student entering college. Many dread mathematics. These should be assured that mathematics is not so difficult, and that it will prove interesting if carefully studied. Father Theobald believes that in the opening session of a freshman course the class should be told that the daily paper exercise is the recognized way toward mastery in mathematics, and that for this reason adequate credit will be allowed for daily work satisfactorily completed.

11. If p_O is the observed proportion of positive differences between the reactions of two organisms to the same treatment or between the effects of two treatments on the same organism, p_E the expected proportion, n the number of observations, N_1 the number of observations necessary for highly significant results (R. A. Fisher's 1% point) and N_5 the necessary number of observations for significance (Fisher's 5% point), the ratio of the difference of the observed proportion and the expected proportion to the standard deviation is

$$\frac{Np_O - Np_E}{\sqrt{Np_Eq_E}}.$$

Professor Brandt makes use of this in testing the significance of the deviation of the observed proportion from the expected. If n is equal to N_1 the deviation is highly significant and if equal to N_5 it is significant.

12. Professor McGaw stressed the difference between two methods of presenting mathematics especially to young students at and before college level. He differentiated these into Static and Dynamic methods. The Static is the usual method in which subjects are taught with the aid of fixed figures and ideas; the Dynamic method makes use of changing figures. The ideas were illustrated with string diagrams, strings being drawn around pegs on a board. Changes in the positions of the pegs, marking points or vertices, were followed by appropriate and visual changes in the relations of the several parts of the figure. Generalization would seem to be favored by the adoption of such an idea in presentation. The applicability of this notion to topics in plane geometry and calculus and to special objects was illustrated.

13. Professor Allen in his paper considers various definitions of probability —psychological (both topological and numerical); axiomatic; definition by "equipossibility"; and by frequency. The latter is preferred, but there are difficulties, particularly in connection with physical applications. Some of these difficulties deal with the existence of the limit of relative frequency in an em-

pirical sequence: possible solutions may be obtained by restriction of the type of admissible sequence, by postulate, or by extension of the definition of probability.

14. In his paper Professor Reilly showed how the Makeham and the "excess" formulas for determining the purchase price of a bond could be generalized for the case when dividends are payable p times a year, and the investment rate is nominal convertible m times a year, $p \neq m$. By the use of a computing machine the computation of the purchase price can be readily found without logarithms.

J. F. REILLY, *Secretary*

THE ANNUAL MEETING OF THE NEBRASKA SECTION

The annual meeting of the Nebraska Section of the Mathematical Association of America was held in conjunction with the annual session of the Nebraska Academy of Sciences at the University of Nebraska in Lincoln, on Friday afternoon, April 28, 1933, with Professor T. A. Pierce of the University of Nebraska as chairman.

Several visitors were present as guests, and the following seventeen members of the Association: M. A. Basoco, A. K. Bettinger, W. C. Brenke, C. C. Camp, A. L. Candy, D. C. Dearborn, J. M. Earl, J. D. Fitzpatrick, M. G. Gaba, A. L. Hill, J. M. Howie, R. M. McDill, G. D. Nichols, T. A. Pierce, Lulu Runge, Mabel F. Schmeiser, R. B. Thompson.

Officers were elected for the ensuing year as follows: Chairman, J. M. Earl, University of Omaha; Secretary-Treasurer, J. M. Howie, Nebraska Wesleyan University; and as member of executive committee, T. A. Pierce, University of Nebraska.

The following papers and reports were presented:

1. "The possible number of magic squares" by Professor A. L. Candy, University of Nebraska.
2. "Dynamical trajectories" by D. C. Dearborn, University of Nebraska.
3. "The absolute regression line" by Professor C. C. Camp, University of Nebraska.
4. "Solutions of certain Diophantine equations" by R. B. Thompson, University of Nebraska.
5. "On a certain class number recurrence of Kronecker" by Professor M. A. Basoco, University of Nebraska.
6. "The generalized rational field" by H. B. Roberts, University of Nebraska.
7. "Polynomials of best approximation over infinite regions" by Professor J. M. Earl, University of Omaha.

Abstracts of some of these papers follow, numbered in accordance with their numbers in the list of titles:

1. This is a continuation of the paper presented by Professor Candy one year ago. The purpose of this further study is to get some idea of the number of approximately symmetric 12×12 squares that can be constructed by the method of "Current Groups."

Starting with the 3×3 square he constructs twelve 6×6 squares, by selecting the groups in 6 different ways, and balancing the rows and columns in slightly different ways, when the numbers in each group have a common difference. From each of these squares 64 different squares can be formed by reversing (interchanging columns) one or more pairs of symmetric groups, or by inverting pairs of symmetric rows. All of these squares are symmetric with respect to the center except the two end numbers in the two middle columns and the two middle rows.

In the case of the 12×12 square the groups can be selected in 33 different ways. By substituting these 33 different sets of groups in the twelve 6×6 squares mentioned above, he has actually written down 151 different 12×12 squares in which the groups in the 6×6 squares have remained unchanged, and the groups in the 12×12 squares are all written in the cyclic order. By reversing (interchanging columns) and inverting (interchanging rows) one or more groups of the fundamental 6×6 squares, (as mentioned above) he gets 9248 12×12 squares in which the order of their groups remain unchanged. Finally, by writing columns of groups in different orders, namely, cyclic and crossed, and reversing one or more pairs of symmetric groups, and inverting double pairs of symmetric groups which are balanced as to rows, he gets a grand total of 249,048,103,172,096.

By varying the order of the groups in the separate columns, as well as in the rows, and disregarding symmetry, this number of possible 12×12 squares can be vastly increased.

2. Professor Kasner, (Transactions of the American Mathematical Society, vol. 7, p. 401) has shown the necessary and sufficient conditions that a triply infinite set of plane curves may be the trajectories due to a positional field of force. This paper is a consideration of the restrictions on an otherwise arbitrary surface such that the system of curves derived by section and orthogonal projection from the surface will satisfy these conditions. It is found that the surface must be a developable and further that it must satisfy a certain partial differential equation. It is possible to find the components of the force field associated with any surface satisfying these two conditions.

3. Ordinarily in statistics one has the two regression lines

$$Y-\overline{Y} = r\frac{\sigma_y}{\sigma_x}(X-\overline{X}) \text{ and } (X-\overline{X}) = r\frac{\sigma_x}{\sigma_y}(Y-\overline{Y})$$

depending on the choice of independent variable. In case one wishes to treat both variables alike one is led to a line such that the sum of the squares of its distances from the original points is a minimum. This gives the absolute regression line

$$Y-\overline{Y}=\frac{\sigma_y^2-\sigma_x^2+\sqrt{(\sigma_y^2-\sigma_x^2)^2+4p^2}}{2p}(X-\overline{X}),$$

where $p=r\sigma_x\sigma_y=\sum(X_iY_i)/n-\overline{X}\,\overline{Y}$. The median regression line which makes the sum of the absolute values of the distances from the line a minimum is unsuitable because it is not always unique. If unique it usually passes through one or more of the original points with about the same number on either side. By holding one point fixed one may easily estimate by using horizontal distances whether rotation to right or left will diminish the sum of the perpendicular distances. Weighted points count as multiple points. However, this line does not fit in so well with ordinary probability theory as the absolute regression line defined above.

4. The purpose of Mr. Thompson's paper was to present parametric solutions of two very general Diophantine equations,

$$X_1^2+X_2^2+X_3^2+\cdots+X_p^2=W^n$$

and

$$X_1^2+a_1X_2^2+a_2X_3^2+\cdots+a_{p-1}X_p^2=Y_1^2+b_1Y_2^2+b_2Y_3^2+\cdots+b_{n-1}Y_n^2.$$

The solutions were obtained directly from the stated problem in terms of independent parameters, thus obtaining multiple infinitudes of solutions, according to the number of parameters involved.

6. This is a field formed by taking one-rowed matrices of three elements each with multiplication $\cdot$, and addition $+$, defined as follows: $(a, b, c)+(x, y, z)=(az+2by+cx, bz+cy, cz)$; and $(a, b, c)\cdot(x, y, z)=(ax, by, cz)$; $(a, b, c)=(x, y, z)$, if $x/a=y/b=z/c$.

7. Professor Earl's paper considers the approximation to a given function of two variables by means of a sequence of polynomials which are determined so as to minimize the integral over an unbounded region of the product of a non-negative weight function and the mth power of the magnitude of the error.

A. L. HILL, *Secretary*

THE TENTH MEETING OF THE INDIANA SECTION

The tenth meeting of the Indiana Section of the Mathematical Association of America was held Friday and Saturday, May 5 and 6, 1933, at Indiana University, Bloomington, Indiana.

There were forty-two registered for the meetings on Saturday including the following twenty-one members of the Association:

Gladys L. Banes, H. T. Davis, W. E. Edington, P. D. Edwards, E. D. Grant, E. H. C. Hildebrandt, F. H. Hodge, E. L. Klinger, Mayme I. Logsdon, Florence Long, Juna M. Lutz, H. A. Meyer, T. W. Moore, J. A. Reising, C. K. Robbins, Fred Robertson, D. A. Rothrock, L. S. Shively, H. E. Slaught, R. O. Virts, K. P. Williams.

On Friday evening an informal reception was held in the East Parlors of the Student Building for arriving members and their guests. At six o'clock a dinner was held in the Grill room of the Indiana Union Building. Professor K. P. Williams acted as toastmaster. Short talks were given by Professor Fernandus Payne, Dean of the Graduate School and head of the department of zoology, and by Professor H. E. Slaught of the University of Chicago. At eight o'clock Professor Slaught gave a public lecture on the subject: "The lag of mathematics behind literature and art in the early centuries." This lecture was later given on the program of the national meeting of the Mathematical Association of America held in Chicago in June and an account of it will appear in the report of that meeting.

The meeting on Saturday was presided over by Professor K. P. Williams of Indiana University, chairman of the Section. At the business meeting the following officers were elected: Chairman, Professor Juna M. Lutz, Butler University; Vice-Chairman, R. O. Virts, Central High School, Fort Wayne; Secretary-Treasurer, Professor P. D. Edwards, Ball State Teachers College.

The retiring chairman's address was given by Professor Williams on the subject, "Early theories of comet orbits." Professor Williams presented the important phases of the early theories and gave especial attention to the very great difficulties involved in the calculation of orbits. Many of the solutions by first rate mathematicians have been all but useless to the practical astronomer because of the extreme difficulty of carrying out the necessary calculations.

The following papers were presented:

1. "Dynamic symmetry" by Professor S. A. Cain, Department of Botany, Indiana University, by invitation.
2. "The Mathematical Association and a decade of mathematics in Indiana" by Professor W. E. Edington, De Pauw University.
3. "Concerning modular functions" by Professor W. E. Maier, Purdue University, by invitation.
4. "Some elementary formulas suggested by an elementary equation in trigonometry" by Professor F. H. Hodge, Purdue University.
5. "The expansion of an arbitrary operative function in successive derivatives" by Professor Fred Robertson, Iowa State College.
6. "Polar line coordinates" by Professor C. K. Robbins, Purdue University.
7. "Some applications of periodogram analysis" by P. W. Overman, Indiana University, by invitation.

Abstracts of the papers follow, the numbers corresponding to the numbers in the list of titles.

1. The term "dynamic symmetry" was coined by Professor Jay Hambidge to describe some of the relationships he discovered in Greek art of the Classical Period. His later researches led to its discovery in ancient Greek writings. It had failed of translation because moderns had no concept of the meaning until Hambidge had developed it, by an analysis of art objects, as a system of pro-

portion probably used consciously by the Greeks in the construction of art and architectural objects. The underlying theme is that of root-rectangular areas and a peculiar area, the whirling-square rectangle, which are manipulated so as to form *commensurate areas* and gnomens by diagonals and perpendiculars—the construction being such as largely to control the "idea" of the art object as to form, and, to a certain extent, ornament. The end and side of the whirling-square rectangle are as 1 to 1.618. This has been called the ϕ ratio and is found in many diversified phenomena. The root rectangles have 1 as their end and $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$, as their sides.

The Greeks probably got the proportions from the ancient Egyptians as the ϕ ratio has been found in the pyramids dating back as far as 4700 B.C. The ϕ ratio rectangle and its reciprocal equal the root-five rectangle, both of which, along with the other root-rectangles, could have been obtained from a square by at least two simple methods of construction used by Egyptians in the "cording of the temple." Why these proportions are "good" in art it is difficult to say, but many modern artists and craftsmen have found it profitable to employ the methods in manipulation of their ideas for designs.

It is truly a remarkable phenomenon that the ϕ ratio should be found as a fundamental underlying theme in the architecture of plants, especially in phyllotaxy, the arrangement of leaves on the stem. Most leaves are arranged in a spiral system the relations of which form a numerical series known as the Fibonacci summation series, running 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, the ratio of which converges on 1.618, the ϕ ratio. This series has been known for centuries but the explanation has only recently been found by A. H. Church of Oxford in continued proportionate growth of leaf primoidia arranged in curve systems on a conic-ovoid growing point. The ubiquitous spiral in nature has been the subject of many interesting speculations and studies and is frequently logarithmic in nature, due in the main to growth phenomena.

The ϕ ratio is also that of Euclid's proposition, to cut a line in extreme and mean proportion, the so-called Golden Section. There have been many formulae of beauty, perfect figures and more or less mystical obsessions held by philosophers of the past and almost any good art will permit of analysis leading to formulae and approaching scientific exactness, but that is certainly neither the whole explanation nor the sole need for creative work. That the ϕ ratio should have been used by the ancient Egyptians and the classical Greeks in the production of good art, constitute the Golden Section, be inherent in the root-five rectangle and other geometrical figures, and constitute one of the most wide spread of botanical phenomena is without any correlating explanation.

2. The history of the Mathematical Monthly and the problems of its publication were sketched briefly in order to show how the problem of its continuation led up to the conception and organization of the Mathematical Association through the combined efforts of the founder of the Monthly, B. F. Finkel, and the three active organizers, H. E. Slaught, E. R. Hedrick and W. D. Cairns. Some original correspondence concerning the charter membership was shown.

The Indiana Section of the Association was reorganized on an active basis on October 16, 1924, at Indianapolis, and has since met regularly in the spring at the various universities and colleges over the state. The programs have always included addresses by speakers of note among whom have been Jacques Hadamard, of Paris, F. R. Moulton, Jacob Kunz, D. W. Moorehouse, Warren Weaver, L. C. Karpinski, T. C. Fry, G. A. Bliss, Cornelius Lanczos, and H. E. Slaught.

The membership of the Section has increased from 30, in 1916, to 65 according to the last directory. A total of 20 research papers by members of the Section have been published in reputable mathematical journals during the past decade. The interest aroused by the meetings of the Section has led to the formation of several undergraduate mathematical clubs in the colleges of Indiana. The advances made in recent years in the development of graduate study in the state was pointed out and the fact that six Ph.D. degrees with mathematics as a major were conferred in the state during the past four years was presented as evidence of general quickening in mathematical interest no little part of which may be attributed to the work of the Indiana Section of the Association.

3. Let $0 < F(\omega)$ and $e^{\pi i\omega} = q$. The classical identity

$$\left(\sum_{v=-\infty}^{\infty} q^{v^2} \right)^2 = 2 \sum_{n=-\infty}^{\infty} \frac{q^n}{1 + q^{2n}}$$

was proved in a new way due to the fact that, generally,

$$\lim_{l \to \infty} \sum_{0<h+k<l} \frac{(-1)^h}{2h + 1 + 2k\omega}$$

$$= \lim_{l \to \infty} \sum_{0<h+k<l} \frac{(-1)^k}{2h + (2k+1)\omega} \cdot \begin{cases} i \\ -i, \end{cases} \text{ if } \begin{cases} 0 < F(\omega) \\ 0 > F(\omega). \end{cases}$$

4. Starting with the identity $\cos 20° \cos 40° \cos 80° = 1/8$ this paper gives several methods of proving this identity with several general formulas together with proofs suggested by the methods of proof for the original identity.

5. The problem is the determination of a method of expanding an arbitrary operative function $f(x, z)$ in a series of successive powers of z. The symbol z shall be interpreted to mean the operational derivative d/dx and its powers the corresponding successive derivatives.

The components z^i define the function $f(x, z)$. These component functions are grouped according to the formula of Schmidt for the discovery of a normalized orthogonal system of functions by linear combinations of the given set.

The function is then expanded in the Fourier manner in terms of these functions $\phi_i^{(z)}$ for the range $0 < z \leqq 1$ of the independent variable. The resulting series is then rearranged in terms of the successive powers of z.

The expansion is stated thus,

$$f(x, z) = \sum_{n=1}^{\infty} A_n(x) \sum_{i=1}^{n} \sum_{\alpha} (-1)^{\beta+i+1} \frac{1}{N_{n\ldots\beta\ldots i}} z^i$$

where

$$A_n(x) = \int_0^1 f(x, z)\phi_i(z)dz$$

and α is every possible combination of $(n-1)\cdots(i+1)$ in the order given, β is the number of indices omitted, $(\beta=0, 1, \cdots, n-i-1)$ and $1/N_i$ is the reciprocal of the square root of the norm if the index is not repeated but is the product of these expressions by $\int_0^1 \phi_i(z)z^i dz$ when the index is repeated.

6. The ordinary polar coordinates of a point also determine a line perpendicular to the radius vector at its end point and may, therefore, be thought of as the coordinates of this line.

To find the polar line coordinate equation of any curve, express its pedal curve in polar coordinates. This type of coordinates proved to be extremely cumbersome in practise but some of the by products turned out to be both new and interesting. For example, if, to three lines through a point perpendiculars of lengths a, c and b are dropped from another point then $c \sin(A+B) = a \sin A + b \sin B$ where A is the angle between b and c, and B the angle between a and c.

7. In this paper the theory of the Schuster periodogram was developed. Applications were shown to such divergent subjects as sun-spots, weather changes, market fluctuations, recurrence of earthquakes, stellar variations, magnetic disturbances, etc. The speaker announced the completion of an elaborate set of tables for the computation of periodograms. Values of $\sin\theta$ and $\cos\theta$, $\theta = 2\pi s/u$, have been computed to 8 decimal places for s from 0 to u, and u from 5 to 75 by integers.

At the afternoon session of the Section a resolution was adopted expressing the appreciation of the members of the Section for the hospitality and courtesy extended to them by the members of the mathematics department of Indiana University.

P. D. EDWARDS, *Secretary*

THE MAY MEETING OF THE MARYLAND—DISTRICT OF COLUMBIA—VIRGINIA SECTION

The May meeting of the Maryland—District of Columbia—Virginia Section of the Mathematical Association of America was held at the University of Virginia on Saturday, May 13, 1933.

Sixty-three persons attended the meeting including the following forty-two members of the Association: O. S. Adams, M. W. Aylor, Archie Blake, J. W. Blincoe, W. E. Byrne, Paul Capron, Orpha A. Culmer, Tobias Dantzig, Alexander Dillingham, J. A. Duerksen, W. H. Echols, Mary Ewin, P. J. Federico,

Isabel Harris, F. E. Johnston, L. M. Kells, W. D. Lambert, A. E. Landry, Gil-, lie A. Larew, B. Z. Linfield, J. J. Luck, Florence M. Mears, Ethel I. Moody Eugenie M. Morenus, F. D. Murnaghan, E. J. Oglesby, E. K. Paxton, W. T. Puckett, K. S. Purdie, O. J. Ramler, Beulah Russell, L. W. Smith, C. M. Sparrow, H. E. Stelson, J. H. Taylor, Mildred E. Taylor, H. W. Tyler, T. L. Wade, F. M. Weida, C. H. Wheeler, G. T. Whyburn, Evelyn P. Wiggin.

The annual Fall meeting will be held at the George Washington University on Saturday, December 2, 1933.

The following officers for the year 1933–34 were elected: Chairman, B. Z. Linfield, University of Virginia; Secretary, F. M. Weida, George Washington University; additional members of executive committee, Abraham Cohen, Johns Hopkins University and Alexander Dillingham, U. S. Naval Academy.

Professor Lipót Fejér, of the University of Budapest, an outstanding figure in the very significant school of mathematics in Hungary, was the invited speaker. He addressed the afternoon session on "The characterization of some remarkable systems of points of interpolation by means of conjugate points, of the Cotes's numbers, and of certain external properties."

The following papers were presented at the morning session:

1. "On the frequency of high correlation coefficients in small samples" by Archie Blake, U. S. Coast and Geodetic Survey.

2. "A problem concerning the homogeneity of continua" by C. H. Wheeler, Johns Hopkins University.

3. "Four-dimensional orthogonal matrices" by Professor F. D. Murnaghan, Johns Hopkins University.

4. "Projective geometry and vector analysis" by Professor Tobias Dantzig, University of Maryland.

5. "A paradox in differential geometry" by Professor B. Z. Linfield, University of Virginia.

6. "The Taylor series construction of a function using only the derivative process" by Professor W. H. Echols, University of Virginia.

7. "Syzygies for Weitzenböck's irreducible complete system of Euclidean concomitants for the conic" by T. L. Wade, University of Virginia.

Abstracts of some of the papers follow, the numbers corresponding to the numbers in the list of titles:

1. Given an integer $n>1$ and a number r such that $|r| \leqq 1$, we inquire what is the probability that two random sets of n numbers each will have a correlation coefficient at least as large in absolute value as $|r|$. We base the discussion not upon the usual procedure of drawing the variates from a normal population, but on the assumption that vectors in all directions in the hyperplane $\sum_{i=1}^{n} x_i = 0$ occur with equal frequency. For statistical problems in which this hypothesis is approximately satisfied, we obtain useful formulas for low as well as high values of n, and not only in the real number-system, but also for complex numbers and quaternions.

To extend the definition of a correlation coefficient to these number-systems, we consider two vectors $\xi \equiv (x_1 \cdots x_n) \neq 0$ and $\eta \equiv (y_1 \cdots y_n) \neq 0$. In the statistical case, $\sum x = \sum y = 0$. We define the right correlation coefficient of η with respect to ξ as the value of the fraction $\bar{\xi}\eta/(M\xi\, M\eta)$, where M denotes the modulus of a vector. This definition leads to an orthogonality property and a least norm property analogous to the orthogonality and least square properties in the real case. We distinguish between right and left correlation coefficients because multiplication is not commutative for quaternions.

2. A continuum M is said to be homogeneous provided that for any pair of points a and b of M there exists a homeomorphism T of M into itself such that $T(a) = b$. In this paper conditions are studied in order that a locally connected continuum M should be "cyclic element homogeneous," i.e., in order that for any two non-degenerate cyclic elements C_1 and C_2 of M there should exist a homeomorphism of M into itself sending C_1 into C_2.

5. It is well known that for a real surface to have an umbilic at a point we must impose two conditions upon the partial derivatives at that point. Still, if in the quadratic equation giving the principal radii at that point we impose the single condition that its roots be equal, the surface will have an umbilic at that point. Professor Linfield pointed out that when the surface is given, the discriminant of the quadratic equation for the principal radii is the product of $(1+p^2)(1+q^2)$ by

$$\begin{vmatrix} 1+p^2 & pq \\ pq & 1+q^2 \end{vmatrix} \left(\frac{r}{1+p^2} - \frac{t}{1+q^2}\right)^2 + (pq)^2 \left(\frac{2s}{pq} - \frac{r}{1+p^2} - \frac{t}{1+q^2}\right)^2,$$

whose vanishing will impose clearly two conditions on the first and second partial derivatives p, q and r, s, t. More generally, if A and B are second order real symmetric matrices and the determinant of one of them is positive, the condition that the roots of $\det(A - tB) = 0$ be equal will always amount to two conditions, and the matrices A and B will then be linearly dependent. For, when $\det(A) > 0$, the discriminant is the product of $A_{11}A_{22}$ by

$$\begin{vmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{vmatrix} \left(\frac{B_{11}}{A_{11}} - \frac{B_{22}}{A_{22}}\right)^2 + A_{12}^2 \left(\frac{2B_{12}}{A_{12}} - \frac{B_{11}}{A_{11}} - \frac{B_{22}}{A_{22}}\right)^2.$$

6. (1). *Prerequisites.* (a) If the derivative of a function is zero in a region C the function is constant in the region. (b) If a series of functions and the series of their derivatives are both absolutely and uniformly convergent in C then the derivative of the sum of the functions is equal to the sum of the derivatives. (c) A bounded infinite assemblage assigns at least one limit-number. In particular, a bounded defined sequence assigns one and only one limit-number.

(2). *Definition.* A function $f(z)$ is regular in a region C when there can be assigned an absolute number M such that

$$\left|\frac{f^{(n)}(z)}{n!}\right| < M^n,$$

M independent of n, for each z in C, $n=0.\ 1,\ 2, \cdots$. Then the bounded sequence $\left|f^{(n)}(z)/n!\right|^{1/n}$ assigns in $(0,\ M)$ a limit α. Then for $1/R=\alpha+\epsilon$, there can be assigned a y such that $y-z<R$; then the sequence

$$\left| (y-z)^n \frac{f^{(n)}(z)}{n!} \right|$$

is summable and the series

$$\sum (y-z)^n \frac{f^{(n)}(z)}{n!}$$

is absolutely and uniformly convergent in a circle with center at z and radius R.

(3). If a is the center and R the radius of convergence of

$$\sum (x-a)^n \frac{f^{(n)}(a)}{n!}$$

then when x is any point in circle R_a and y is any point in the ellipse, foci a and x and major axis R, the series

$$\phi(y) \equiv \sum (x-y)^n \frac{f^{(n)}(y)}{n!}$$

for all values of y is absolutely and uniformly convergent in the ellipse and its derivative as to y is zero in the ellipse and the series is constant in this region. The points a and x are in the ellipse. Therefore $\phi(x)=\phi(a)$ or

$$f(x) = \sum (x-a)^n \frac{f^{(n)}(a)}{n!}$$

for any assigned x in the circle R_a.

7. By means of the fundamental identities of the symbolic method an algebraically complete set of nine syzygies are found connecting the eighteen irreducible concomitants of the conic, as established by Dr. Roland Weitzenböck in *Über Bewegungsinvarianten*, Wiener Berichte 122 (1913) and 123 (1914). One syzygy is on invariants and covariants, two on invariants and contravariants, and the remaining six on all types of concomitants.

F. M. WEIDA, *Secretary*

A PHOTO-ELECTRIC NUMBER SIEVE

By D. H. LEHMER, Altadena, California

Until recently, the mathematician has been considered the only scientist fortunate enough not to need any laboratory equipment to carry out his researches. The last decade, however, has shown a tendency among mathematicians to adapt the existing commercial calculating machines to their computa-

tions,[1] and in rare cases to invent devices of their own to perform special operations.[2]

There is an important class of problems in the theory of numbers which cannot be solved readily by any commercial calculating machine, so that the number-theorist has had to resort to a kind of graphical method (similar to the celebrated Sieve of Eratosthenes) in handling problems of this sort. By the generous cooperation of the Carnegie Institution of Washington it has been possible to construct a new kind of calculating machine, applicable to this class of problems, in which modern physics has made its contribution to the oldest and least practical branch of mathematics.

In order to make clear the details of this machine, it is desirable to say a few words about the kind of problems it can solve and the underlying principles by means of which it solves them. We shall not attempt to describe completely this class of problems, since this would require the introduction of various concepts and notations unfamiliar to the general reader who is not well acquainted with the theory of numbers. It will suffice to illustrate with the following problem which is a typical representative.

Let it be required to find an integer x for which ax^2+bx+c is a square number y^2. Here a, b, and c are given integers and determine the problem.[3] Simple as this problem seems at first sight, it contains as special cases problems of extreme difficulty. Thus $a=1$, and $b=0$ gives $c=y^2-x^2=(y-x)(y+x)$. This problem, then, is equivalent to factoring the number c and is one of the central problems of the theory of numbers. From the fact that a, b, and c are small we must not conclude that x is small. In fact, when $a=1549$, $b=0$, and $c=1$, we find the smallest positive value of x to be

$$x = 12223\ 09542\ 82674\ 74959\ 34242\ 68334\ 63805$$
$$08818\ 07626\ 31786\ 81966\ 09867\ 28279\ 63220.$$

To give an elementary explanation of the method by means of which the machine solves such problems, we may introduce the idea of a finite arithmetic which deals only with the numbers $0, 1, 2, 3, \cdots, p-1$. We may perform addition, subtraction and multiplication in the ordinary manner, but in every case the result is divided by p and the remainder alone preserved. In this way we remain inside our system of numbers. The number of multiples of p which we discard in reducing the answer to a number in our system is as immaterial to us as the number of complete revolutions of the wheel is to the roulette player.

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[1] Dr. L. J. Comrie of Great Britain's Nautical Almanac Office is an exponent of this procedure. See Monthly Notices of the Royal Astronomical Society, 92, (1932), 523–541.

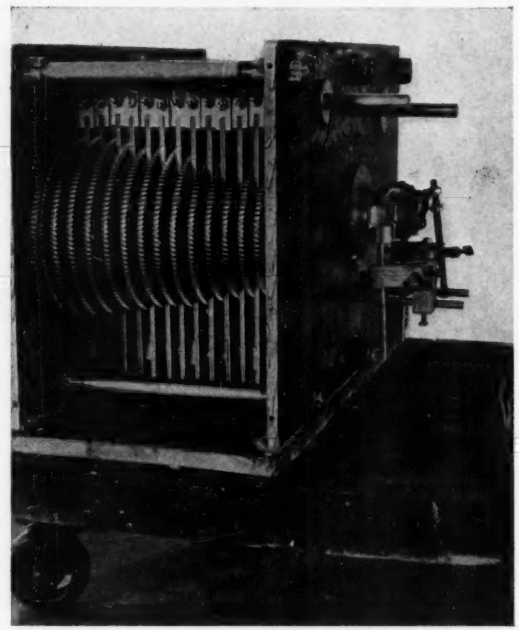
[2] The mechanical integraphs recently developed by Dr. V. Bush of the Massachusetts Institute of Technology may be cited in this connection. See Journal of the Franklin Institute, vol. 212, (1931), 447–488.

[3] The problem is not solved by showing that x exists, nor by giving inequalities which x must satisfy. The problem is to exhibit x.

Let us imagine a problem in ordinary arithmetic requiring the solution in integers of a certain equation. This equation may be translated bodily into our finite arithmetic by merely replacing the coefficients of the equation by their remainders on division by p. To solve the problem in the finite arithmetic is simple enough because the values of the unknown are restricted to lie among the p numbers $0, 1, 2, \cdots, p-1$. If p is small we can find all the solutions by actual trial. But why is it useful to consider a real problem in one of these artificial arithmetics? What interpretation can we give to the solutions of the problem in such a system? To answer these questions let us consider one of the answers to the original problem. If we divide this answer by p, the remainder will be one of the solutions of the problem in the finite arithmetic. The desired answer then is some one of these artificial answers plus a certain multiple of p. This is not much information, but it is easy to obtain. Moreover we have not committed ourselves to the choice of p. In fact, we may choose as many different p's as we like, and if they have no common factor, the information offered by each arithmetic will be independent. The combination of these various bits of information leads to the solution of the original problem. It is this combination which the machine is designed to effect.

In order to obtain a mechanical picture of the situation let us imagine a disk gear with p teeth having a small hole opposite each tooth. These holes are at a constant distance from the periphery of the gear and are numbered from 0 to $p-1$. Let us plug up all the holes except those which correspond to answers to our problem considered in the finite arithmetic. If a light is set behind the gear, then, as the gear rotates, it will transmit the light when and only when the number of teeth turned past has a chance of satisfying the problem as far as this particular gear is concerned. To get the combined effect of using several p's, we may set up several gears parallel to each other. If the gears are mounted so that they have a common line of tangency and are driven at the same linear speed, then the light which shines through a gear will be transmitted or blotted out by the next gear. In this way the gears are allowed to pass judgment upon the eligibility of each number as it turns past. The decision of each judge is not influenced by the rulings of the other judges and his rejection of a candidate is final. When a number is a solution of our problem, however, there will be no dissention among the judges and the beam of light will succeed in running the gauntlet. It is true that other numbers may be thus unanimously elected without being answers to our problem. If enough gears are used, however, these undesirables can be eliminated altogether. In the kind of problems that the machine solves about one half of the holes are stopped up in each gear, and there are 30 gears. This means that we can expect an extraneous alignment of holes about once in $2^{30} = 1073741824$ numbers. By this time the reader has grasped the essential features of the mechanical system. It is clear that the gears may be driven at great speed and still there will be ample time for the light to pass through every gear when the answer arrives. In fact this time interval is one ten-thousandth of a second.

So much for the mechanical system. All that is needed now, is an alert and untiring eye behind the last gear to observe and report the appearance of the tiny flash of light. This eye is a photo-electric cell. The energy produced by the small amount of radiation falling on the retina of the cell is very small indeed. It must be sufficiently magnified to operate a circuit breaker for the electric motor driving the machine. This is done by means of a six stage amplifier and a three stage relay system. The amplifier magnifies the energy from the photo-electric cell about 700 million times and transmits it to a delicate vacuum tube relay, the first of a series of three which finally stop the machine.



Courtesy of the Carnegie Institution

View of the mechanical system showing one series of gears.

To give a concrete example of how the machine works, let us consider the following problem: to find a value of x for which

(1) $$91894770302976x^2 + 287722528867021824x + 256527596541064768$$

is a perfect square. These large coefficients are not arbitrary numbers. In fact they arise quite naturally in an investigation into the possible factors of the Mersenne number $2^{79}-1$. The numbers 2^n-1 where n is a prime have been the subject of investigation since the time of Euclid. Twelve of these numbers have been proved prime and twelve composite ones have been completely factored. Since 1924 it has been known that if a value of x exists for which (1) is a

square then $0<x<39110012$. This problem was considered in each of the finite arithmetics corresponding to a prime or a power of a prime $p<127$, and the appropriate holes in the corresponding gears were stopped up. This presents the problem to the machine, which, canvassing numbers at the rate of 300000 a minute, can cover the above range for x in about two hours without attention. As a matter of fact the power was automatically shut off in 12 seconds, and the machine coasted to a stop. Reversing the machine slowly and substituting the human eye for the photo-electric cell, the light was seen to shine through at $x=56523$ according to the reading on the revolution counter. Substituting this value of x in (1) we obtain at once the number 309853160646773276521024, which is the square of 556644555032. Hence our problem is solved. Incidentally this leads to the factorization

$$2^{79}-1 = 2687\cdot 202029703\cdot 1113491139767.$$

We take this opportunity to answer a few questions that are almost always asked by those inspecting the machine. The first of these has to do with the period of the machine. Since each gear goes through the same position over and over again, the same is true of the machine as a whole so that the light will pass through the machine periodically. This is quite true in theory. The period of the machine is clearly the least common multiple of the periods of the gears. This means that the machine will return to its original position after

2497 19431 65929 91015 26347 12970 31949 33696 87002 72210

teeth have turned past. Even though the machine runs at the speed of 300000 teeth per minute it would wear out long before it got really started on its period.

Another question that is always asked is: Do you know approximately what the answer is before you start? In the problems we consider, the unknown is lost among millions of consecutive numbers. If we could tell in advance just which million contained the answer it would not have been necessary to construct the machine.

As a final question: What is the exact relation between the machine and mathematics itself? We do not hesitate to apply mathematics to physics, but in applying physics to mathematics there arises the question of the reliability and accuracy of the results. Fortunately, in this case, this question is not a vital one. The accuracy of the mechanical system is absolute. The machine does not depend upon measurements or estimated values. The gears arrive inexorably at their appointed positions with absolute certainty. The photo-electric system, however, with its enormous amplification and its elaborate safeguards against outside disturbances, would not have been possible a few years ago, and even to-day it is easy for the unintiated to underestimate its reliability. This system is indispensable, however, since any substitute for the weightless beam of light could not be depended upon at the high speeds that some problems demand.

After all, such questions are for the technician. As for the mathematician, he is glad to obtain immediately verifiable answers to his problem no matter

how unreliable their source may be. For what sources are more unreliable and yet more indispensable than imagination, intuition or inspiration?

The successful completion of the machine has been made possible first of all by the Carnegie Institution of Washington, which once more came to the aid of the number-theorist by a grant of the necessary funds. We also wish to acknowledge the assistance of Mr. T. J. Palmateer of Stanford University for valuable advice in constructing the mechanical parts of the machine. We are also much indebted to Dr. R. C. Burt of Pasadena who not only constructed the photo-electric system but, on account of his interest in the project, has generously given space in his laboratory where the machine is now operating.[1]

LAGRANGE'S QUINTIC FOR THE NEUTRAL HELIUM ATOM

By W. E. COX, JR., Tulane University

In a paper investigating the straight line solutions of the problem of three bodies, Lagrange[2] arrived at a quintic equation, the real solutions of which determined the relative positions of the body along the straight line. The purpose of this paper is to discuss the corresponding quintic for the case of the neutral helium atom.[3]

Let the mass of the nucleus be m_2 and the masses of the electrons m_1 and m_3. Assume a positive charge, e_2, on m_2 and negative charges, $-e_1$ and $-e_3$, on m_1 and m_3 respectively. Let ξ_i, η_i, ζ_i $(i=1, 2, 3)$ be the coordinates of the three bodies with the center of gravity of the three masses as the origin and with axes rotating uniformly in the $\xi\eta$-plane with angular velocity ω. Let r_{ij} $(i, j=1, 2, 3)$ denote the distance between m_i and m_j. Then the following differential equations follow from Coulomb's law,

$$(1)\qquad \begin{aligned} &\frac{d^2\xi_i}{dt^2} - 2\omega\frac{d\eta_i}{dt} - \omega^2\xi_i - \frac{k^2}{m_i}\frac{\partial U}{\partial \xi} = 0,\\ &\frac{d^2\eta_i}{dt^2} + 2\omega\frac{d\xi_i}{dt} - \omega^2\eta_i - \frac{k^2}{m_i}\frac{\partial U}{\partial \eta} = 0,\\ &\frac{d^2\zeta_i}{dt^2} - \frac{k^2}{m_i}\frac{\partial U}{\partial \zeta} = 0, \quad i = 1, 2, 3, \end{aligned}$$

where

$$U = \frac{e_1e_2}{r_{12}} - \frac{e_1e_3}{r_{13}} + \frac{e_2e_3}{r_{23}}.$$

We seek a particular solution of these equations in which the three masses lie in a straight line and rotate in circles about the center of gravity of the sys-

[1] Since the above was written the apparatus has been put on exhibition at A Century of Progress.

[2] Lagrange's *Collected Works*, vol. VI, p. 277.

[3] See H. E. Buchanan, this MONTHLY, vol. 38 (1931), pp. 511–521.

tem. The coordinates of the three bodies are, then, constants and their derivatives zero. Equations (1), therefore, become

$$
\begin{aligned}
&-m_1\omega^2\xi_1 - \frac{k^2e_1e_2(\xi_2-\xi_1)}{r_{12}^3} + \frac{k^2e_1e_3(\xi_3-\xi_1)}{r_{13}^3} = 0,\\
&-m_2\omega^2\xi_2 - \frac{k^2e_1e_2(\xi_1-\xi_2)}{r_{12}^3} - \frac{k^2e_2e_3(\xi_3-\xi_2)}{r_{23}^3} = 0\\
&-m_3\omega^2\xi_3 + \frac{k^2e_1e_2(\xi_1-\xi_3)}{r_{13}^3} - \frac{k^2e_2e_3(\xi_2-\xi_3)}{r_{23}^3} = 0,\\
&-m_1\omega^2\eta_1 - \frac{k^2e_1e_2(\eta_2-\eta_1)}{r_{12}^3} + \frac{k^2e_1e_3(\eta_3-\eta_1)}{r_{13}^3} = 0,\\
&-m_2\omega^2\eta_2 - \frac{k^2e_1e_2(\eta_1-\eta_2)}{r_{12}^3} - \frac{k^2e_2e_2(\eta_3-\eta_2)}{r_{23}^3} = 0,\\
&-m_3\omega^2\eta_3 + \frac{k^2e_1e_3(\eta_1-\eta_3)}{r_{13}^3} - \frac{k^2e_2e_3(\eta_2-\eta_3)}{r_{23}^3} = 0,\\
&-\frac{e_1e_2(\zeta_2-\zeta_1)}{r_{12}^3} + \frac{e_1e_3(\zeta_3-\zeta_1)}{r_{13}^3} = 0,\\
&-\frac{e_1e_2(\zeta_1-\zeta_2)}{r_{12}^3} - \frac{e_2e_3(\zeta_3-\zeta_2)}{r_{23}^3} = 0,\\
&+\frac{e_1e_3(\zeta_1-\zeta_3)}{r_{13}^3} - \frac{e_2e_3(\zeta_2-\zeta_3)}{r_{23}^3} = 0.
\end{aligned}
\tag{2}
$$

The last six of equations (2) are satisfied by $\eta_i=\zeta_i=0$ $(i=1, 2, 3)$. We will have found a solution of equations (1) in which the masses lie in a straight line along the ξ-axis if we find constants ξ_1, ξ_2, ξ_3, which satisfy the first three of equations (2) with $\eta_i=\zeta_i=0$. Assume $\xi_3>\xi_2>\xi_1$; since for any position of the nucleus other than between the two electrons there are no solutions as can be ascertained by going through the following operations for the other two cases. The unit of distance is arbitrary, so let $r_{12}=\xi_2-\xi_1=1$. Then equations (2) become

$$-m_1\omega^2\xi_1 - k^2e_1e_2 + \frac{k^2e_1e_3}{(\xi_3-\xi_1)^2} = 0 \tag{3}$$

$$-m_2\omega^2(\xi_1+1) + k^2e_1e_2 - \frac{k^2e_2e_3}{(\xi_3-\xi_2)^2} = 0 \tag{4}$$

$$-m_3\omega^2\xi_3 - \frac{k^2e_1e_3}{(\xi_3-\xi_1)^2} + \frac{k^2e_2e_3}{(\xi_3-\xi_2)^2} = 0. \tag{5}$$

Equation (5) may be replaced by

$$m_3\xi_3 + m_2\xi_1 + m_2 + m_1\xi_1 = 0, \tag{6}$$

obtained by adding equations (3), (4), and (5).

Solving equations (3) and (6) for ω^2 and ξ_3 in terms of ξ_1 and substituting in equation (4), the following equation in ξ_1 is obtained,

$$\begin{aligned}
&e_1e_2M^4(m_1+m_2)\xi_1^5 + e_1e_2M^3(m_2M + 4m_2^2 + 2m_2m_3 + 4m_1m_2 + 2m_1m_3)\xi_1^4 \\
&\quad + M^2[e_1e_2(m_1+m_2)(m_2+m_3)^2 + 4e_1e_2m_2^2M + 2e_1e_2m_2m_3M \\
&\quad + 4e_1e_2m_2(m_1+m_2)(m_2+m_3) + e_1e_3m_2^2(m_1+m_2) - e_3m_3^2(e_1m_2+e_2m_1)]\xi_1^3 \\
&\quad + m_2M[e_1e_2(m_2+m_3)^2(M+2m_2+2m_1) + 2e_1e_2m_2(m_1+m_2)(m_2+m_3) \\
&\quad + 4e_1e_2m_2M(m_2+m_3) + e_1e_2m_2^2M - e_3m_3^2(e_1M+2e_2m_1) \\
&\quad - 2e_1e_3m_3^2(m_2+m_3)]\xi_1^2 + m_2[2e_1e_2m_2M(m_2+m_3)^2 \\
&\quad + e_1e_2m_2(m_1+m_2)(m_2+m_3)^2 + 2e_1e_2m_2^2M(m_2+m_3) \\
&\quad - e_1e_3m_3^2(m_2+m_3)^2 - e_2e_3m_1m_2m_3^2 - 2e_1e_3m_3^2M(m_2+m_3)]\xi_1 \\
&\quad + e_1m_2(e_2m_2^2 - e_3m_3^2)(m_2+m_3)^2 = 0,
\end{aligned} \tag{7}$$

where

$$M = m_1 + m_2 + m_3.$$

Since this is a fifth degree equation in ξ_1, there is at least one real root. The coefficients of ξ_1^5 and ξ_1^4 are definitely positive and the constant term is $e_1m_2(e_2m_2^2 - e_3m_3^2)(m_2+m_3)^2$. Thus, if $e_3m_3^2 > e_2m_2^2$, there is at least one real positive root. Similarly, there is at least one real negative root if the constant term is positive. In the case of the helium atom $e_2 = 2e_3$ and m_2 is much greater than m_3, and hence there is at least one real negative root.

If we let $A = \xi_3 - \xi_2$ and substitute the resulting value of ξ_1,

$$\xi_1 = -\frac{m_3A + m_2 + m_3}{M},$$

in equation (7), a fifth degree equation in A is obtained,

$$\begin{aligned}
&e_1e_2m_3(m_1+m_2)A^5 + e_1e_2m_3(3m_1+2m_2)A^4 \\
&\quad + m_3(e_1e_2m_2 + 3e_1e_2m_1 - e_2e_3m_1 - e_1e_3m_2)A^3 \\
&\quad + m_1(e_1e_2m_3 + e_1e_3m_2 - 3e_2e_3m_3 - e_2e_3m_2)A^2 \\
&\quad - e_2e_3m_1(3m_3+2m_2)A - e_2e_3m_1(m_2+m_3) = 0.
\end{aligned} \tag{8}$$

This corresponds to Lagrange's quintic for the case under consideration. A definite distribution of the three bodies along the ξ-axis, and, therefore, a definite particular solution of our problem, is given by every real positive root of equation (8). It is seen that the coefficients of A^5 and A^4 are both positive and the coefficient of A and the constant term are both negative, but that the coef-

ficients of A^3 and A^2 may be either positive or negative for arbitrary values of e_i and m_i; and, in particular, the coefficient of A^3 may be negative and that of A^2 positive, as, for example, when $e_1=4$, $e_2=1$, $e_3=5$, $m_i=1$. In any case there is an odd number of changes of sign in the coefficients and, hence, at least one real positive root. In every case, except that in which the coefficient of A^3 is negative and the coefficient of A^2 positive, there is only one change of sign in the coefficients and, therefore, only one real positive root. In the particular case in which the coefficient of A^3 is negative and that of A^2 positive there are three changes of sign in the coefficients and, hence, a possibility of three positive roots. The author has been unable to determine whether or not there can actually be three positive roots of this equation.

In the case of the helium atom, $m_3=m_1=m$ and $2e=e_2=2e_1=2e_3$. Making these substitutions in (8), we obtain

$$(9)\quad 2(m+m_2)A^5+2(3m+2m_2)A^4+(4m+m_2)A^3 - (4m+m_2)A^2-2(3m+2m_2)A-2(m+m_2)=0.$$

There is but one change of sign in the coefficients of this equation, and therefore, there is only one real positive root. It is obvious that this root is $A=1$, which was to be expected from the way in which the unit of measurement was chosen.

QUESTIONS, DISCUSSIONS, AND NOTES

EDITED BY R. E. GILMAN, Brown University, Providence, Rhode Island

The department of Questions and Discussions in the Monthly is open to all forms of activity in collegiate mathematics, including the teaching of mathematics, except for specific problems, especially new problems which are reserved for the department of Problems and Solutions.

QUESTION CONCERNING THE MAXIMUM TERM IN THE DIATOMIC SERIES—PROPOSED[1] BY A. A. BENNETT

A REPLY BY ALFRED BRAUER, University of Berlin

Given k, let π_k denote the continued product of the first k primes, p_i, starting with $p_i=2$. For example, $\pi_5=2.3.5.7.11=2310$. Let p_0 be 1.

Question: *Given k, what is the maximum number of consecutive integers each of which is divisible by one of the primes $p_1, p_2, \cdots, p_k$?*

This question had already been studied by Legendre in his work *Théorie des Nombres*[2] when he tried to prove the known theorem (first proved by Dirichlet) that there exist an infinite number of primes in any arithmetical progression $mx+n$, $(m, n)=1$. Legendre showed, as did Mr. Bennett, the existence of at

[1] This MONTHLY, vol. 39 (1932), pp. 27–28.

[2] Legendre, *Théorie des nombres*, Tome II, Paris (1830), pp. 71–79.

least $2p_{k-1}-1$ consecutive integers each of which is divisible by some factor of π_k. But Legendre thought this number also to be the maximum number of the problem in question. This however is incorrect, though the fallacy was for a long time unnoticed.

Dirichlet was the first who recognized the incorrectness of the proof Legendre gave for his statement,[1] but the statement itself he considered to be true and tried in vain to find a valid proof for it.

In 1858, the Academy of Paris proposed as a problem for competition:[2] *To prove the statement of Legendre or to find the actual maximum number of the problem.* None of the papers delivered contained a satisfactory solution of the question. The prize was awarded to A. Dupré who showed[3] that the statement of Legendre is correct for $p_k \leqq 19$ and for $p_k = 29, 31, 41$, but that it does not hold generally because it is incorrect for $p_k = 23, 37$ and for $43 \leqq p_k \leqq 113$.

Later on, Moreau[4] and Piltz[5] independently pointed out the incorrectness of the statement of Legendre. Piltz supposed the statement to be wrong for all sufficiently great values of p_k.

Recently, Mr. H. Zeitz and I gave the first general proof[6] that *the statement of Legendre is incorrect for all primes* $p_k \geqq 43$. Morever, given any small number $\epsilon \leqq 0$, there is always a number $k_0 = k_0(\epsilon)$ such that for all integers $k > k_0$ there exist

$$N = (4 - \epsilon)p_k$$

consecutive integers each of which is divisible by one of the primes $p_1, p_2, \cdots, p_k$.

This result was improved in a paper of E. Westzynthius[7] who for the above number N substitutes the value

$$N = (2 - \epsilon)e^C \frac{\log\log p_k}{\log\log\log p_k} p_k,$$

where C denotes the constant of Euler.

---

[1] Dirichlet, *Beweis des Satzes, dass jede unbegrenzte arithmetische Progression, deren erstes Glied und Differenz ganze Zahlen ohne gemeinschaftlichen Faktor sind, unendlich viele Primzahlen enthält*, Werke I, pp. 313–342.

[2] *Rapport sur le concours pour le grand prix de sciences Mathématiques*, Comptes Rendus 48 (1859), pp. 487–488.

[3] Dupré, *Examen d'une proposition de Legendre relative à la théorie des nombres, ouvrage placé en première ligne par l'Académie des Sciences dans le concours pour le grand prix de Mathématiques de 1858*, Paris (1859) Mallet-Bachelier.

[4] Extrait d'une lettre de M. Moreau, Nouvelles Annales (2) XII (1873), pp. 323–324.

[5] Piltz, *Ueber die Häufigkeit der Primzahlen in arithmetischen Progressionen und über verwandte Gesetze*, Habilitationsschrift Jena 1884.

[6] A. Brauer und H. Zeitz, *Ueber eine zahlentheoretische Behauptung von Legendre*, Sitzungsberichte d. Berliner Mathematischen Gesellschaft, 29 (1930), pp. 116–125.

[7] E. Westzynthius, *Ueber die Verteilung der Zahlen, die zu den n ersten Primzahlen teilerfremd sind*, Commentationes Physico-Mathematicae, Societas Scientiarum Fennicae, vol. 25 (1931).

A NOTE ON THE ROOTS OF A CUBIC

By E. C. KENNEDY, College of Mines, El Paso, Texas

If the cubic

$$X^3 + AX^2 + BX + K = 0$$

(A, B, and K rational) has one or more rational roots it may often be solved very quickly by the scheme illustrated below. If c is a rational root and $a \pm \sqrt{b}$ are roots, a and b rational, it can be shown that

$$2a + c = -A \tag{1}$$

$$b = K/c + a^2 \tag{2}$$

$$2ac^2 - Bc - K = 0. \tag{3}$$

From (3), $D = B^2 + 8Ka$ must be made a perfect square by some rational value of a. This almost instantly gives us the three roots of our cubic. To illustrate the brevity of our method let us examine, for integral roots,

$$X^3 - 304X - 1920 = 0 \tag{4}$$

Here,

$$2a + c = 0 \tag{5}$$

$$b = -1920/c + a^2 \tag{6}$$

$$D/4 = 23104 - 3840a \tag{7}$$

To find a we arrange the work as follows:

$$\begin{array}{r} 23104 \\ 3840 \\ \hline 19264 \\ 3840 \\ \hline 15424 \\ 3840 \\ \hline 11584 \\ 3840 \\ \hline 7744 = \text{a perfect square.} \end{array}$$

Thus $a = 4$. From (5) $c = -8$ and from (6) $b = 256$, and the roots of the cubic are $X_1 = 4 + 16 = 20$, $X_2 = 4 - 16 = -12$, $X_3 = -8$.

This scheme is often much quicker than finding the roots by trial, especially if the constant term has a large number of factors as does 1920. The method will always work if the roots are integers—though a may be negative or zero (the latter if $A = K/B$).

Rational roots present no difficulty. For example, to solve

$$6X^3 - 21X^2 + 2X - 56 = 0, \tag{8}$$

$$2a + c = 21/6$$

$$b = -56/c + a^2$$

$$D = (2/6)^2 - \frac{8(56)a}{6} = \frac{4 - 224a'}{36}. \qquad (a' = 12a)$$

Evidently $a' < 0$ (otherwise c would be complex since a' is an integer) so we write

$$\begin{array}{r} 4 \\ \underline{224} \\ 228 \\ \underline{224} \\ 452 \\ \underline{224} \\ 676 \end{array} = \text{a perfect square.}$$

Thus $a' = -3$, $a = -3/12$, $c = 4$, $b = -109/48$. This gives us immediately all three of the roots. We set a' equal to $12a$ instead of equal to $6a$ because the equation might conceivably be resolved into two factors one of which is a quadratic with the coefficient of X (in the quadratic) odd.

An interesting fact may be observed at this stage. By Descartes' Rule there are either three or one positive real root of (8). There are no negative real roots. Now if there are three real roots, it follows that $a > 0$. But we must have $a < 0$ in order to make $4 - 224a > 0$ which it must be in order for c to be real. Therefore, (8) has exactly one real root. This is quicker than computing the discriminant and determining the number of real roots in that way.

As a final illustration let us test

$$X^3 - 14X^2 + 9X - 12 = 0. \tag{9}$$

Here $a > 0$ and we write mechanically $D = 81 - 48a$, $81 - 48 = 33$, not a perfect square. Therefore no rational roots.

Note: The scheme described above will always work and is often quite short. It seems that every method for finding rational roots has some drawbacks. In our method we cannot always determine the sign of a immediately, though we may often find very restricted limits for it. Again it sometimes happens that we obtain a value of a that will not serve. In such a case we quickly eliminate that possibility by considering (1), (2), or (3). If the roots are all rational then there are three acceptable values of a, any one of which will serve. The scheme has one big advantage over the usual method in that the amount of labor involved is wholly independent of the number of factors in the constant term. It is not expected to displace other methods of determining rational roots, but it is often useful.

LINE REPRESENTATION OF THE HYPERBOLIC FUNCTIONS

By C. A. HUTCHINSON, University of Colorado

The following construction for representing the values of the hyperbolic functions by lines may be of interest. That part of it which concerns the hyperbolic sine and cosine is old.

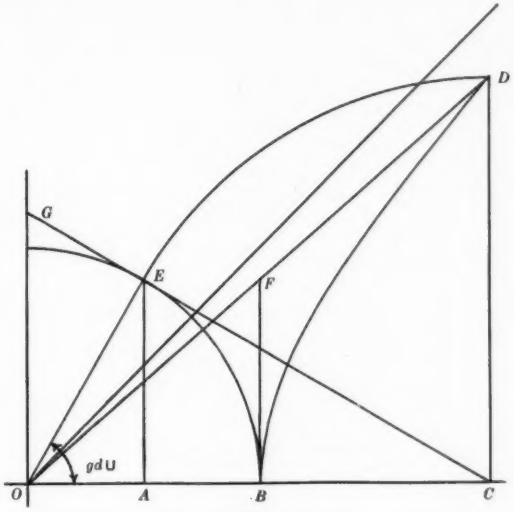


Fig. 1

Draw the "unit" equilateral hyperbola

$$x^2 - y^2 = 1,$$

and the corresponding circle

$$x^2 + y^2 = 1.$$

Let D be any point on the hyperbola, and

$$u = (2 \times \text{area of hyperbolic sector } OBD)/(OB)^2.$$

Draw DC perpendicular to the transverse axis OC. From C draw CG tangent to the circle at E. This can be done easily by intersecting the circle at E by a circular arc with C as center and CD as radius; or E can be located as the intersection of the circle with the line joining D to the other vertex of the hyperbola, H (not shown in Fig. 1). OG, AE and BF are perpendicular to OC. Then

$\sinh u = CD = CE$;	$\coth u = OG$;
$\cosh u = OC$;	$\operatorname{sech} u = OA$;
$\tanh u = AE = BF$;	$\operatorname{csch} u = EG$;

angle $AOE = \text{gd } u = $ gudermannian angle of u.

Note the relations read from right triangles:

from triangle OCE: $\sinh^2 u+1=\cosh^2 u$;

from triangle OAE: $\operatorname{sech}^2 u+\tanh^2 u=1$;

from triangle OEG: $1+\operatorname{csch}^2 u=\coth^2 u$.

The same diagram lends itself also to the representation of the functions of $u/2$ (see Fig. 2). Let the line DH intersect OP at J. Draw JK parallel to OC,

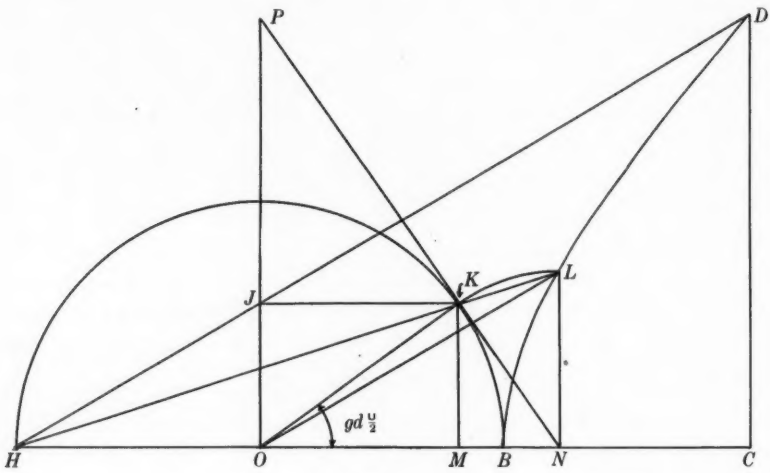


Fig. 2

intersecting the circle at K. At K, draw NP tangent to the circle. Through K and H, draw HL, intersecting the hyperbola at L. Then

$$\sinh u/2=NL=NK; \qquad \coth u/2=OP;$$
$$\cosh u/2=ON; \qquad \operatorname{sech} u/2=OM;$$
$$\tanh u/2=OJ=MK; \qquad \operatorname{csch} u/2=PK;$$
$$\text{angle } NOK=\operatorname{gd} u/2.$$

It will be noticed that line OL gives a bisection of the hyperbolic sector OBD. This line also bisects angle AOE, or gd u. This fact gives an even easier construction of point L. (To make the second figure clearer, the unnecessary lines of Fig. 1 have been omitted.)

Note. After sending the manuscript of this note to the MONTHLY, the author received a copy of Franklin's *Differential Equations for Electrical Engineers*, and noted that it contains a very similar construction for some of the hyperbolic functions.

RECENT PUBLICATIONS

EDITED BY R. A. JOHNSON, Brooklyn College of the City of New York

All books for review should be sent directly to the editor of this department, at Brooklyn College, 66 Court Street, Brooklyn, N. Y., and not to any of the other editors or officers of the Association.

NEW BOOKS RECEIVED

Introductory Mathematics. By J. W. Lasley and E. T. Browne. New York, McGraw-Hill, 1933. xiv+440 pages. $2.75.

The Geometry of Repeating Design and Geometry of Design for High Schools. By A. Day Bradley. New York, Teachers College, 1933. vi+132 pages. $1.50.

Plane Trigonometry. By J. A. Northcott. New York, Long and Smith, 1933. x+150 pages. $1.60; with tables, $2.40.

Plane Trigonometry. By William L. Hart. New York, D. C. Heath, 1933. vi+186 +16 pages, tables iv+124 pages. $2.00.

Plane Trigonometry. By Cecil A. Ewing. New York, McGraw-Hill, 1933. xii+166 pages. $1.20.

McGraw-Hill Five-Place Logarithmic and Trigonometric Tables. Edited by Ralph D. Beetle. New York, McGraw-Hill, 1933. viii+132 pages. 80 cents.

College Algebra. By J. B. Rosenbach and E. A. Whitman. Boston, Ginn and Company, 1933. xii+394 pages. $2.00.

College Algebra. By Raymond W. Brink. New York, The Century Company, 1933. xviii+446 pages. $2.25.

Handbook of Mathematical Tables and Formulas. Compiled by R. S. Burington. Sandusky, Ohio, Handbook Publishers, Inc., 1933. $2.00; $1.25 to students and instructors.

Differential Equations for Electrical Engineers. By Philip Franklin. New York, John Wiley, 1933. viii+300 pages. $2.75.

Contributions to the Calculus of Variations, 1931–1932. Theses submitted to the Department of Mathematics of the University of Chicago, together with a paper by G. A. Bliss and M. R. Hestenes. University of Chicago Press, 1933. x+524 pages. $3.00.

The Fourier Integral and Certain of its Applications. By Norbert Wiener. Cambridge University Press, 1933. xii+202 pages. $3.25.

A First Course in College Mathematics. By William E. Anderson. New York, Harper, 1933. x+326+92 pages. $2.75.

Continuous Groups of Transformations. By Luther P. Eisenhart. Princeton University Press, 1933. x+302 pages. $4.00.

Differential Equations. By Max Morris and O. E. Brown. New York, Prentice-Hall, 1933. xiv+410 pages. $2.75.

REVIEWS

Triumph der Mathematik. Hundert berühmte Probleme aus zwei Jahrtausenden mathematischer Kultur. By Heinrich Dörrie. Breslau, Ferdinand Hirt, 1933. vii+386 pages.

This is a remarkable and unique book. The author has selected from the entire field of mathematics one hundred interesting and significant problems and gives a complete elementary solution for each, elementary in the sense that no use is made of the calculus. The nature and unusual scope of these problems will be clear from a brief summary.

The first section of arithmetic problems includes preliminary tid-bits such as Archimedes' cattle problem, Newton's cattle problem, Berwick's problem of the seven sevens, Bachet's weights problem, and Kirkman's school-girls problem. Then after a clever introduction of the infinite series representing the elementary functions, there are proofs of the fundamental theorem of algebra, of the impossibility of algebraic solution of the general algebraic equation of degree greater than four, and of the transcendence of e and of π.

The second section of problems in plane geometry includes synthetic proofs for the Euler line, nine-point circle, Malfatti's problem, Apollonius's problem of the construction of circles tangent to three given circles, analytic proofs of the impossibility of the trisection of the angle and duplication of the cube, and a construction of the regular seventeen-sided polygon.

The third section deals with conics and cycloids. Included in the twenty-four problems of this section are various construction problems, Steiner's theorem that the envelope of the Wallace lines of a triangle is a three-cusped hypocycloid, the determination of the curvature of a conic, quadrature of the conics, and rectification of the parabola. The chapter ends with a closely-knit condensed treatment of projective geometry including Pascal's and Brianchon's theorems, and applications to the construction of conics.

The fourth section deals with problems in three-dimensional geometry. It includes a few properties of tetrahedra, the determination of the five regular solids, a proof that any quadrilateral can be projected into a square, stereographic projection, and Mercator's projection.

The fifth section deals with nautical and astronomical problems such as the length of a loxodrome on a sphere, the determination of position at sea, duration of polar night, construction of sun-dials, and the retrograde motion of the planets.

The sixth and final section deals with problems of extrema. As typical problems we may cite Fagnano's problem of the triangle of minimum perimeter inscribed in an acute-angled triangle, the maximum brilliance of Venus, determination of shortest twilight, and Steiner's theorem that of isoperimetric curves the circle has maximum area.

There is, of necessity, throughout the book, a certain amount of development, so that reference to earlier problems is frequent, and many problems do

not stand alone. The proofs are, however, in the main those by the great mathematicians whose names are associated with the problems. The extent to which reference to the literature should be made was probably a difficult question for the author; as it is for the reviewer. In general the very brief historical references are confined to author, nationality, and dates; original memoirs are sometimes cited but not in the majority of cases. The reviewer would have preferred to see an exact reference to an original paper for each problem, and a carefully selected, but not exhaustive, group of references which would allow the interested reader to approach the subject from other points of view.

The references given are, in the main, accurate. But problem 36 on the seventeen-sided polygon contains an error. Gauss did prove the possibility of construction of a regular polygon of n sides where n is a prime Fermat number, but he did not give a proof for the impossibility for all other primes, as stated by Dörrie. Here, for example, the single reference to the *Disquisitiones arithmeticae*, 1801, seems totally inadequate for such a historically significant problem. In problem 57 on the rectification of the parabola, credit should be given to Huygens.

The author has high standards of accuracy and rigor which are maintained throughout. It is of course difficult to say what theorems may be considered as well-known and assumed without reference. The reviewer would prefer in the proof of the transcendence of e to have a definite statement that the number of primes is infinite, a theorem necessary for the proof as given. Again in the fundamental theorem of algebra, page 107, it is questionable whether the existence of the minimum should be assumed without comment.

The proof-reading is of the highest order, the only slip in the entire book which was noted being in the tenth decimal place of the approximation to e given on page 52, where 1 should replace 2. A word should be said with regard to the *sans serif* type employed. This may prove a satisfactory type for mathematical work if contrasting type is used for letters in equations. This is not done, and some pages (page 277, for example) are harder reading than is necessary.

This book should be widely circulated. Members of mathematics clubs will here find excellent material for presentation at meetings. It should be read as much for pleasure as for profit. We are under obligations to the author for this discriminating selection of the brilliant problems of mathematics. It is regrettable that there is nothing like it in English.

B. H. Brown

Philosophischer Versuch über die Wahrscheinlichkeit, a translation of Essai philosophique sur les Probabilités, by P. S. de Laplace, edited by R. v. Mises. (No. 233 Ostwald's Klassiker der exakten Wissenschaften). Akademische Verlagsgesellschaft M.B.H., 1932. vii+211 pages.

Two previous translations into German of Laplace's well known essay have appeared, one in 1819 based on the original third edition, the second in 1886

based on the fourth edition. Both editions have long been out of print. The present translation is by Dr. Heinrich Löwy and follows the older translation of 1886 and the fifth edition (1825) of the original. The new translation presents the essay in more modern German, corrects slips and gives more precision to many statements.

The main value of this edition lies in the forty pages of "Anmerkungen" by H. Pollaczek-Geiringer. In general these notes are of two kinds. First, many of the more difficult mathematical passages of the original are explained and put into more modern symbolism. Second, the ideas of Laplace are compared with those of the more modern theory of probability. These comparisons are subjective, and, von Mises being editor of the book, we are not surprised to find in these "Anmerkungen" a very clear presentation of the relative frequency theory and the usual arguments centering about "Gleichmöglichkeit."

A. R. CRATHORNE

Arithmetic for Teachers. By Harriet E. Glazier. New York, McGraw-Hill Book Company, 1932. xv+291 pages. $2.00.

This book might well be called *Elementary Arithmetic from an Advanced Standpoint*. Subject, rather than method, is emphasized throughout the volume.

The author traces the development of arithmetic as an outgrowth of simple natural conditions involving number ideas, through stages of crude number symbolism, into our present scheme of notation and processes of operations with these symbols. The basic principles underlying the fundamental operations are set forth. The book is written as a text for teachers courses in colleges and universities and serves this purpose admirably well. An attractive feature is the list of readings and exercises at the end of each chapter. The typographical set-up of the book is unusually good.

Chapters II and VII should be of interest to the general reader. Chapter II deals with the number concept, clearly explains number bases and number systems, and summarizes the history of notation. Chapter VII, which is on measurement, gives the essentials of a good unit of measure and compares the different standard systems. In Chapters VIII and IX the subjects of geometry and algebra are introduced. The entire book is rich in historical references.

MAY M. BEENKEN

Introduction to Practical Astronomy. By Dinsmore Alter. New York, Thomas Y. Crowell Company, 1933. VIII+80 pages. $2.00.

This book might well be called a laboratory manual, since it treats the material as experiments with very little theoretical discussion. The book is divided into four sections beginning with "preliminary experiments" which include experiments with the vernier, systems of coordinates, and lenses. Section II considers time, relations between coordinate systems, experiments concerning the planets and precession. The greater part of Section III is devoted to the de-

termination of positions on the earth's surface using the engineers transit. Section IV takes up a method of determining the errors of adjustment of an equatorial telescope and the right ascension and declination of objects. There are a liberal number of sheets of coordinate paper in the back of the book.

The experiments cover a wider field than I believe can be found in other elementary books of this nature. It seems however that additional material might profitably be added in the first two sections. A set of good star maps would also be desirable. The sections on the engineer's transit and the equatorial telescope should prove especially interesting for those students who do not expect to take more than an elementary course in astronomy.

The chief criticism of the book is the rather frequent use of poorly constructed sentences especially in the introduction. The material on the whole is well selected and the book will be a welcome addition to the rather limited number of manuals of laboratory astronomy.

M. F. JORDAN

C. A. Bjerknes, sein Leben und seine Arbeit. By V. Bjerknes. Aus dem Norwegischen ins Deutsche übersetzt von E. Wegener-Köppen. Berlin, Julius Springer, 1933. iv+218 pages. RM 8.60.

In this biography V. Bjerknes gives a sympathetic picture of the difficulties and struggles encountered by a young gifted man without means desirous of becoming a scientist. C. A. Bjerknes, well known for his hydrodynamical investigations, was born in Norway in 1825. The economic situation both in his family and in the country was none too good at the time. The School of Mines afforded the only opportunity for scientific study and the school required active service in the mines even for students wanting to specialize in more abstract subjects like mathematics and astronomy. After completing his studies Bjerknes had to accept a position as overseer in the copper mines and the study of mathematics had to be confined to his spare moments. After several disappointments he obtained a small position at the university and shortly afterwards a fellowship for foreign study. He came to Göttingen in 1855 and worked with Lejeune-Dirichlet, to whom he had been recommended, and also attended the lectures of Riemann, where Dedekind and Schering were the only other students. Next year he continued his studies in Paris with Cauchy, Liouville, Bertrand and Lamé.

Upon his return to Norway Bjerknes was appointed professor at the University and started the long series of investigations and experiments, which he continued for the rest of his life, on hydrodynamical phenomena and their analogy to electricity.

The book gives a quaint narrative of the life of a typical scientist. It is a biography of a father by his son, it consequently eulogizes his life and works and is adorned by a large number of personal details.

OYSTEIN ORE

MATHEMATICS CLUBS

EDITED BY F. M. WEIDA, The George Washington University, Washington, D. C.

All reports of club activities, suggestions and topics for club programs, and material of interest to clubs should be sent to F. M. Weida, The George Washington University, Washington, D. C. All manuscript should be typewritten with double spacing, and with margins at least one inch wide.

CLUB ACTIVITIES

1932–1933

THE PI MU EPSILON MATHEMATICAL FRATERNITY

Pi Mu Epsilon is an academic fraternity in institutions of university grade. Its primary aim is the advancement of mathematics and scholarship. It is a living, active, working fraternity of scholars in which the members are actively engaged in study and research and the exchange of ideas in the field of mathematical science.

Pi Mu Epsilon of the University of Alabama

Our chapter has forty active members. On October 11, 1932, five were initiated and on April 25, 1933, seven were initiated.

The meetings and programs were as follows:

September 27, 1932: "The minimum of a definite integral with respect to one sided variation" by Julian D. Mancill.

October 25, 1932: "The four color problem" by William Sell.

November 29, 1932: "Infinite products" by Edna Jane Cofield.

January 31, 1933: "Some properties of a set of polynomials whose zeros are all real" by H. S. Thurston.

February 28, 1933: "Fourier's theory" by G. N. Carmichael.

March 28, 1933: "On the properties of a certain class of collineation groups" by Fred A. Lewis.

April 25, 1933: "Analytic proofs of some theorems concerning a Triape and its related circles" by Boyce Garrett: "Some properties and applications of continued fractions" by Mary Catherine Granade; "Some special roulettes" by Winston M. Scott.

In addition to the above meetings, two socials were given: on December 8, 1932, a bridge party and on May 5, 1933, the annual picnic.

The following members of Pi Mu Epsilon were elected to Phi Beta Kappa this year: Sarah Bloodworth, Elizabeth Shirley, Jane Ward.

One of our members, Mr. William Sell, has revived the Newtonian Club, Freshman Mathematics Society, and reports a very successful year.

The Chapter, this year, has voted to use some of its funds in subscribing to mathematical magazines for use in the general library.

WILLIAM F. ADAMA, *Secretary*

Pi Mu Epsilon of Brooklyn College

The Delta Chapter of Pi Mu Epsilon in New York was installed at Brooklyn College of the College of the City of New York on February 6, 1933. The installation was held at the Hotel Pierrepont, Brooklyn, and was conducted by Professor Lao G. Simons assisted by Professors Marie Whalen and Mina S. Rees, all of the Hunter College Chapter (Beta of New York). Fifteen instructors and thirty-three graduates and undergraduates were initiated. The chapter grew out of the Delta Society which was organized in June, 1931, and held regular meetings since that time. Miss Alma M. Hauland, who has been President of the Delta Society, presided at the initiation banquet.

The officers of the new chapter are: Professor Roger A. Johnson, Director; Miss Elsie Kardonsky, Vice Director; Miss Irene Silverstein, Secretary.

During the first semester, the meetings of the Delta Society were devoted to the study of differential geometry. In the present semester, geometrical transformations have been studied.

R. A. JOHNSON, *Director*

LOCAL MATHEMATICS CLUBS

The Euclid Club of the College of William and Mary

The officers for the year 1932–1933 were: C. S. Sherwood, III, President; Louise Lang, Vice President; Elizabeth Toler, Secretary; Ethel Hartman, Treasurer; Dr. J. M. Stetson and Miss Beulah Russell, Faculty Advisers.

The aim of the club is to stimulate a creative interest in mathematics. Membership is open to all persons who intend to major or minor in mathematics and who have completed nine semester hours and are registered for at least three hours more. A grade of 91 is required in one course and an average of 85 in the other courses. This year we had forty members.

The meetings and programs were as follows:

October 21, 1932: "The influence of mathematics on civilization." Papers were read by Douglas Matthew, Elizabeth Potterfield, Susie Brittle and Eleanor Berger.

November 18, 1932: "The structure of the aeroplane wing" by Dr. Stetson.

December 16, 1932: "Simple aspects of Einstein's theory of relativity" by Alfred Armstrong.

February 17, 1933: "Probability" by Marianne Norris and Douglas Matthew.

March 17, 1933: Business meeting.

April 21, 1933: Election of officers.

April 26, 1933: Banquet. Dr. J. A. C. Chandler was our guest and speaker of the evening.

ELIZABETH TOLER, *Secretary*

The Mathematics Club of Brown University

The meetings and programs were as follows:

November 7, 1932: Scott Read Chatterton, '33, presiding; "Centennial dates" by Mary Helena Quirk, '34; "The puzzler's perplexities" by Charles Roland Eddy, '35.

December 12, 1932: Tina Codianni, '33, presiding; "Parabolic fit" by Vernon Franklin Kenyon, '35; "Areas by machine" by Cyril Garbutt Sargent, '33.

January 16, 1933: Dean Richardson, presiding; "When stars shone in the afternoon" by Professors Currier and Smiley.

February 20, 1933: Garland Balch Russell, '33, presiding; "Brilliant points" by William Fuller Branch, '34; "The path of the pendulum" by Walter Harris Porter, '34.

March 20, 1933: Elizabeth Alma Partridge, '33; presiding; "Archimedes" by Harriet Amy Legg, '34; "Omar Khayyam" by Jean Esther Smith, '33; "Galileo" by Mary Stella Barao, '34.

April 17, 1933: Professor Gilman, presiding; "Repeating decimals" by Professor Widder of Harvard University.

Our committee on program was: Professor Bennett; Professor Oakley; Mary Stella Barao, '34; Scott Read Chatterton, '33; Charles Roland Eddy, '35; Mary Helena Quirk, '34.

Our committee on arrangements was: Mr. Tucker; William Fuller Branch, '34; Edgar Gage Hotaling, '35; Edith Viola Janson, '34; Rhoda Madden, '35.

R. C. ARCHIBALD, *Professor of Mathematics*

The Mathematics Club of the University of Buffalo

The mathematics club of the University of Buffalo is open to all students interested in mathematics whether they are majors or not. The attendance at the meetings this year averaged about thirty. Some of the meetings were social gatherings at the home of the professors and others were business meetings at which student papers were read.

The officers for the year 1932–1933 were: John Wrench, President; Lois Plummer, Vice President; Charlotte Houck, Secretary and Treasurer.

The meetings and programs were as follows:

October 1932: Opening supper. Motion pictures of "The Span Supreme."

November 1932: The club was entertained at the home of Professor and Mrs. Pound with a mathematical spelling bee and games.

January 1933: "A sketch of Einstein" by John Wrench; "A review of 'The Queen of the Sciences' by E. T. Bell" by Alice Link.

March 1933: "Systems and properties of axioms" by Charles Strobel; "Axes and centers of symmetry" by Robert R. Lyle. (The latter was taken from the thesis submitted by Mr. Lyle in partial requirement for the degree of Master of Arts, granted February, 1933.

April 1933: The club was entertained at a Monte Carlo party at the home of Professor and Mrs. Harrington.

May 1933: The retiring president, John Wrench, was host to the club at his home. The Wilfred H. Sherk Memorial Prize in Mathematics was awarded to Charles Strobel for his paper submitted on "Axioms and postulates and their properties."

HARRIET F. MONTAGUE, *Department of Mathematics*

PROBLEMS AND SOLUTIONS

EDITED BY B. F. FINKEL, OTTO DUNKEL, H. L. OLSON, AND W. F. CHENEY, JR.

ELEMENTARY PROBLEMS

Send all communications about Elementary Problems and Solutions to Wm. Fitch Cheney, Jr. Dept. Box 35, Storrs, Connecticut.

The Department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. Problems may be submitted unaccompanied by their solutions.

PROBLEMS FOR SOLUTION

E 48. *Proposed by Norman Anning, University of Michigan.*

If the squares of the sines of a set of angles are in harmonic progression, show that the squares of the tangents of the same angles are also in harmonic progression.

E 49. *Proposed by Arnold Dresden, Swarthmore College.*

In the following problem in long division it is required to determine the non-zero digits, which are designated by x's. The zeros are properly shown.

$$
\begin{array}{l}
x\,0\,x\,0\,x)x\,x\,x\,0\,x\,0\,x\,0\,0\,x\,0(x\,0\,x\,0\,0\,x \\
\quad x\,x\,0\,x\,0\,x \\
\quad\quad\overline{} \\
\quad\quad\quad x\,0\,x\,x\,0 \\
\quad\quad\quad x\,0\,x\,0\,x \\
\quad\quad\quad\overline{} \\
\quad\quad\quad\quad x\,0\,x\,0\,x\,0 \\
\quad\quad\quad\quad x\,0\,x\,0\,x\,0 \\
\quad\quad\quad\quad\overline{} \\
\quad\quad\quad\quad\quad x\,0\,0\,0
\end{array}
$$

E 50. *Proposed by H. T. R. Aude, Colgate College.*

Two fractions, F_1 and F_2, when written in a certain scale of notation, are $0.3737\cdots$ and $0.7373\cdots$ respectively. When written in a second scale of notation, these same fractions are $0.2525\cdots$ and $0.5252\cdots$ respectively. Find the fractions and the bases used for the two scales.

E 51. *Proposed by J. Rosenbaum, The Milford School, Milford, Conn.*

Prove that there are just three pairs of non-negative integers (x, y), which satisfy the equation, $3\cdot 2^x+1=y^2$, and determine them.

E 52. *Proposed by Moshe Abrahami, Case School of Applied Science.*

Find the area of a triangle in terms of the altitude, interior angle bisector, and median, all from the same vertex of the triangle.

E 53. *Proposed by Morgan Ward, California Institute of Technology.*

If s is a positive integer, prove that

$$D_x^s(x^s \ln x) = s!(\ln x + 1 + \tfrac{1}{2} + \tfrac{1}{3} + \cdots + 1/s).$$

SOLUTIONS

E 15 [1932, 607]. *Proposed by Mrs. Pearl C. Miller, Washington University.*

Prove that if two external angle-bisectors of a scalene triangle are equal, then the sines of the three interior half-angles form a geometric progression. By external angle-bisector is here meant that segment of the line bisecting the exterior angles at a vertex of a triangle, intercepted between that vertex and the opposite side of the triangle.

Solution by Roy MacKay, Albuquerque High School.

Let $AU=s$ bisect the exterior angle A of the triangle ABC, and $BV=t$ bisect the exterior angle at B. Then angle $AUC=(B-C)/2$ and angle $BVA=(C-A)/2$. Now by the law of sines, $s=b \sin C/\sin \frac{1}{2}(B-C)$ and $t=c \sin A/\sin \frac{1}{2}(C-A)$. Equating $s=t$, and $b/c=\sin B/\sin C$ gives

$$\sin A \sin \tfrac{1}{2}(B - C) = \sin B \sin \tfrac{1}{2}(C - A).$$

By a series of easy reductions, there now results

$$2 \sin \tfrac{1}{2}A \cos \tfrac{1}{2}A \sin \tfrac{1}{2}(B - C) = 2 \sin \tfrac{1}{2}B \cos \tfrac{1}{2}B \sin \tfrac{1}{2}(C - A),$$

$$\sin \tfrac{1}{2}A\{-2 \sin \tfrac{1}{2}(B+C) \sin \tfrac{1}{2}(B-C)\} = \sin \tfrac{1}{2}B\{-2 \sin \tfrac{1}{2}(C+A) \sin \tfrac{1}{2}(C-A)\},$$

$$\sin \tfrac{1}{2}A(\cos B - \cos C) = \sin \tfrac{1}{2}B(\cos C - \cos A).$$

Changing the cosines to one minus twice the square of the sine of the half-angle, rearranging and dividing out $\sin \frac{1}{2}A+\sin \frac{1}{2}B$, which is not zero, we have the desired result:

$$\sin \tfrac{1}{2}A \sin \tfrac{1}{2}B = \sin^2 \tfrac{1}{2}C.$$

Solved also by Simon Vatriquant and the proposer.

E 22 [1933, 490]. *Proposed by R. M. Winger, University of Washington.*

As a western version of problem E 7, find the digits represented by the various letters in the following problem in addition, and determine whether or not the solution is unique. (Except that obviously R and L and S and G are interchangeable.) No two different letters represent the same digit.

$$\begin{array}{cccc} S & E & N & D \\ M & O & R & E \\ G & O & L & D \\ \hline M & O & N & E & Y \end{array}$$

Solution by Simon Vatriquant, Athénée Royal d'Ixelles, Brussels, Belgium.

1. Obviously $M=1$ or 2. If $M=2$, since at most $S+G=17$, $O=0$ or 1. If $O=0$, then in the second column E would have to be 9, which would reduce $S+G$. If $O=1$, the carry from the second to the first column would be insufficient. Consequently, $M=1$.

2. The sum of the ten digits is 45, and the sum of the digits of the three numbers to be added $\equiv M+O+N+E+Y \pmod 9$.

3. The alternate sum of the digits of the three numbers to be added $\equiv M-O+N-E+Y \pmod{11}$.

4. $2D+E \equiv Y \pmod{10}$. Hence D is neither 0 nor 5.

Therefore we may write

$$2Y + N + 1 = D, \text{ or } D + 9, \text{ or } D + 18.$$

$$3D + 4E + 4O \equiv N + 2 \pmod{11}.$$

Cancelling the values which give a repeated digit, there remain the following systems: (Note that the word *MONEY* is determined in each case and we may thus calculate the sums $S+G$ and $R+L$)

Y	N	D	E	O	$R+L$	$S+G$	R,L	S,G
0	5	6	8	3	11	11	4,7	2,9
0	5	6	8	3	11	11	2,9	4,7
0	7	8	4	6	5	14	2,3	5,9
2	9	3	6	0	impossible			
6	3	7	2	5	8	13	0,8	4,9
7	0	6	5	2	impossible			
6	9	4	8	0	8	9	3,5	2,7
7	5	2	3	0	impossible			
8	4	3	2	0	impossible			
8	5	4	0	7	5	15	2,3	6,9
9	5	6	7	4	impossible			
	6	7	5	0	impossible			

Permuting the R, L and S, G in all possible ways gives twenty-four different solutions, of which the six fundamental ones are

2856	4856	9478	9237	2894	6054
1348	1328	1624	1502	1038	1720
9376	7396	5638	4587	7054	9734
13580	13580	16740	15326	10986	17508

Solved also by M. A. Heaslet.

E 23. *Proposed by R. K. Morley, Worcester Polytechnic Institute.*

In the following sum and product, the digits from 1 through 9 are represented in some order by the letters A through I. Determine the representation and prove that it is unique.

$$AB = CD + EF, \; EF = G \times HI.$$

Solution by W. E. Buker, Leetsdale, Pa.

1. Since $EF = G \times HI$, $G \neq 1$ or 5. $I \neq 1$ or 5.

2. Since $AB < 99$, $11 < CD$, it follows that $EF < 87$. But the solution $98 = 12 + 86$ is ruled out by the repeated 8, and $EF \neq 85$ since neither G nor $I = 5$. Hence $EF < 85$.

3. Since the problem demands that no two letters have the same value, we are reduced to the following possible values for G and HI.

$G=2$, $HI=17, 18, 19, 34, 38, 39$
$G=3$, $HI=16, 18, 19, 26$
$G=4$, $HI=13, 17, 18, 19$
$G=6$, $HI=13$
$G=7$, $HI=12$

Trial of these values shows that the only set which completely fulfills the conditions of the problem are $G=4$, $H=1$, $I=7$. Consequently, the only solution is $93=25+68$ and $68=4\times17$.

Solved also by Simon Vatriquant and the proposer.

E 24 [1933, 111]. *Proposed by R. K. Morley, Worcester Polytechnic Institute.*

There are just three proper fractions with denominators less than a hundred which may be reduced to lowest terms by illegitimately canceling a digit. One of these is

$$\frac{26}{65} = \frac{2\not{6}}{\not{6}5} = \frac{2}{5}.$$

Find the other two and confirm the statement that there are no others.

Solution by Pincus Schub, Gratz College, New York City.

If, after cancelling the digit, n, from numerator and denominator, the resulting proper fraction in lowest terms is a/b, then the original fraction must have had one of these four forms:

(A) $(10a+n)/(10b+n)$, (B) $(10n+a)/(10n+b)$, (C) $(10n+a)/(10b+n)$, or (D) $(10a+n)/(10n+b)$. Here a, b and n are each positive integers <10, and $a<b$.

It is easily seen that (A) and (B) each lead to a contradiction. (C) is also impossible, since if $a/b=(10n+a)/(10b+n)$, then $n=9ab/(10b-a)$, $<a$. This last is true because $9b<10b-a$. From this it would follow that $b=an/(10n-9a)<0$, which is impossible. Hence only (D) remains possible.

From (D), $n=9ab/(10a-b)$, which is larger than $9ab/(10a-a)=b$. That is, $b<n$. Again from (D), $a=bn/[9(n-b)+n]\leqq\frac{1}{2}b$, since $9(n-b)\geqq n$. Hence $a<5$.

Since $b=10a-9ab/n$, and $a<b$, it follows that n is a multiple of 3. This leaves just four pairs of values for a and n to make b an integer, namely: 1,6; 1,9; 2,6; and 4,9. The resulting fractions are 16/64, 19/95, 26/65 and 49/98. But the last is not reduced to lowest terms by the cancellation, and hence there are only three proper fractions with denominators less than a hundred which may be reduced to lowest terms by illegitimately cancelling a digit.

Solved also by W. E. Buker, L. G. Butler, L. S. Johnston, Theodore Lindquist, E. Mrock, C. W. Munshower, Simon Vatriquant and the proposer.

E 25. *Proposed by W. F. Cheney, Jr., Connecticut State College.*

Find the only ten-place integer which is both a square and a triangular number.

Solution by H. T. R. Aude, Colgate University.

The desired number must be of the form $\frac{1}{2}n(n+1)$ to be triangular, and must also be equal to some square, m^2. Replacing $2n$ and $2m$ by $x-1$ and y, respectively, gives the Pellian equation: $x^2=2y^2+1$. Its primary solution is $x=3$, $y=2$, which through the relation

$$x+y\sqrt{2}=(3+2\sqrt{2})^k, \qquad [k=0, 1, 2, 3, \cdots]$$

yields all positive integral solutions. We next examine the values of k to see which, if any, will make m^2 a ten-place integer. The only value is $k=7$, whence $y=80782$, and the desired number is

$$1{,}631{,}432{,}881=40{,}391^2=(57{,}121)(57{,}122)/2.$$

Solved also by W. E. Buker, W. R. Ransom, Simon Vatriquant and the proposer.

Advanced Problems

Send all communications about Advanced Problems and Solutions to B. F. Finkel, Springfield, Mo. All manuscript should be typewritten, with double spacing, and with margins at least one inch wide.

Problems containing results believed to be new, or extensions of old results are especially sought. The editorial work would be greatly facilitated if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known textbooks or results found in readily accessible sources, will not be proposed as problems for solution in the Monthly. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

PROBLEMS FOR SOLUTION

3624. *Proposed by N. A. Court, University of Oklahoma.*

If the lines joining the vertices of a tetrahedron to its circumcenter meet the respective opposite faces in the centroids of these faces, each edge of the tetrahedron is equal to its opposite edge.

3625. *Proposed by Robert E. Moritz, University of Washington.*

Show that

$$\sum_{k=1}^{n}(-1)^{k+1}{}_nC_k k^n = (-1)^{n+1}n!$$

for all positive integral values of n.

3626. *Proposed by Ruth G. Mason, Berkeley, California.*

Let all letters represent integers, $k>1$, $x\geqq 0$, $P>1$; and let $C_k(x)$ denote the sum of k consecutive integers, the smallest of which is $x+1$. Find the necessary and sufficient conditions that there exists a value for x such that $C_k(x)=P$.

3627. *Proposed by W. H. Rasche, Virginia Polytechnic Institute.*

Given a cone of revolution and two ellipses E^h and F^v; by the processes of descriptive geometry so fix the position of the cone relative to the H- and V-planes of projection that E^h will be the H-base of the cone and F^v its V-base. Show that there is no solution unless the sum of the acute angles which the axis of the cone makes with H and V is equal to or less than 90°.

3628. *Proposed by Mannis Charosh, New Utrecht High School, Brooklyn, N. Y.*

M_1, M_2, M_3 are the midpoints of sides A_2A_3, A_3A_1, A_1A_2 of a triangle $A_1A_2A_3$. B_1, B_2, B_3 are the symmetrics of any point P with respect to M_1, M_2, M_3. Prove the following theorems:

(*a*) A_1B_1, A_2B_2, A_3B_3 are concurrent in some point Q which bisects each of these lines.

(*b*) Triangles $A_1A_2A_3$ and $B_1B_2B_3$ are congruent and their corresponding sides are parallel.

(*c*) If P be taken as the orthocenter of $A_1A_2A_3$, Q will be its circumcenter.

(*d*) If P be taken as the circumcenter of $A_1A_2A_3$, Q will be the center of its nine point circle.

(*e*) If N_1, N_2, N_3 are the midpoints of sides B_2B_3, B_3B_1, B_1B_2 respectively, then A_1N_1, A_2N_2, A_3N_3 are concurrent in some point R.

(*f*) M_1N_1, M_2N_2, M_3N_3 are concurrent in Q, which bisects each of them.

(*g*) P, Q, R are collinear, and Q bisects PR.

(*h*) If M is the centroid of $A_1A_2A_3$ and N is the centroid of $B_1B_2B_3$, then M and N lie on the line PQR, and Q bisects MN.

(*i*) If P describes any locus, Q and R describe similar loci.

3629. *Proposed by J. Rosenbaum, Milford, Conn.*

Find a formula for the sum

$$\sum_{x=1}^{n} x(x+1)(x+2)\cdots(x+p).$$

Note by the Editors. Find also the sum where the general term is multiplied by a^x, $a \neq 1$.

3630. *Proposed by Rufus Crane, Ohio Wesleyan University.*

It is known that there are three spheres escribed to the tetrahedron each of which corresponds to a pair of opposite edges. Let the tetrahedron be $A_1A_2A_3A_4$, let the area of the face opposite to A_i be α_i, and let the dihedral angle whose edge is A_iA_j be δ_{ij}. Prove that the sphere corresponding to the pair of edges A_iA_j and A_kA_l lies external to the edge A_iA_j if $\alpha_k+\alpha_l<\alpha_i+\alpha_j$, a condition which may be reduced to $\alpha_k\alpha_l \cos^2(\delta_{ij}/2)<\alpha_i\alpha_j \cos^2(\delta_{kl}/2)$.

SOLUTIONS

3500 [1931, 340]. *Proposed by Emma M. Gibson, Springfield High School, Springfield, Mo.*

Show that the primitive of the differential equation, $p^2(1-x^2)=(1-y^2)$ is $x^2+y^2-2Axy=1-A^2$ and derive this equation by taking the sine of the sum of two angles, both of which are arcsines.

See solution by J. D. Leith [1932, 430].

Note by Editor Finkel. This problem appears on page 47, Ex. 7 (ξ), of Forsyth's *Treatise on Differential Equations*, and in the form

$$\frac{dy}{dx}+\left(\frac{1-y^2}{1-x^2}\right)^{1/2}=0,$$

on page 19, Ex. 5, of the same text. On page 19 the author presents two methods of solution, one leading to $\sin^{-1}y+\sin^{-1}x=c$, and the other, on integrating by parts, leading to $y(1-x^2)^{1/2}+x(1-y^2)^{1/2}=c$. Leith's method of obtaining the primitive in the required rational form was to write it as $\sin^{-1}y=\sin^{-1}x+\sin^{-1}k$, and then take the sine of both sides of the equation. By taking the cosine of both sides of the equation $\sin^{-1}y-\sin^{-1}x=\sin^{-1}k$, one obtains $(1-x^2)^{1/2}(1-y^2)^{1/2}+xy=(1-k^2)^{1/2}$; then by transposing xy, squaring, and substituting A for $(1-k^2)^{1/2}$, the required form is obtained quite readily.

But if one takes the sine of both sides of the equation $\sin^{-1}y+\sin^{-1}x=\sin^{-1}k$, a little more difficulty is encountered. The sole purpose Miss Gibson had for proposing this problem was to exhibit this particular method of deriving the rational form of the primitive. The problem should have been stated by saying "derive this equation by taking the sine of the sum of the two angles

$\sin^{-1} y$ and $\sin^{-1} x$." By this procedure, we obtain $y(1-x^2)^{1/2}+x(1-y^2)^{1/2}=k$. Squaring both sides,

$$y^2 - x^2y^2 + 2xy\sqrt{(1 - x^2)(1 - y^2)} + x^2 - x^2y^2 = k^2,$$

or

$$(x^2 + y^2 - k^2) - 2x^2y^2 = -2xy\sqrt{(1 - x^2)(1 - y^2)}.$$

Squaring again,

$$(x^2 + y^2 - k^2)^2 - 4x^2y^2(x^2 + y^2 - k^2) + 4x^4y^4 = 4x^2y^2 - 4x^4y^2 - 4x^2y^4 + 4x^4y^4.$$

Whence

$$(x^2 + y^2 - k^2)^2 = 4x^2y^2(1 - k^2),$$

and therefore

$$x^2 + y^2 - k^2 = \pm 2xy\sqrt{1 - k^2}.$$

Letting $A = \pm(1-k^2)^{1/2}$ and transposing, one obtains

$$x^2 + y^2 - 2Axy = 1 - A^2$$

as required.

3566. [1932, 490] *Proposed by V. F. Ivanoff, San Francisco, Calif.*

A perpendicular CM_0 is drawn to the line segment M_0M_n and the points $M_1, M_2, \cdots, M_{n-1}$ are taken on M_0M_n so that $\angle M_0CM_1 = \angle M_1CM_2 = \cdots = \angle M_{n-1}CM_n = \alpha$. By parallel displacement of the vectors $CM_i (i=2, 3, \cdots, n)$ a broken line $CM_1M_2'M_3' \cdots M_n'$ is formed with $M_1M_2' = CM_2$ and $M_{i-1}'M_i' = CM_i (i=3, 4, \cdots, n)$. Find the locus of the vertices of this broken line if $CM_0 \to 0$ and $\alpha \to 0$ in such a way that $CM_0/\alpha = K$, K constant.

Solution by Otto Dunkel, Washington University.

Take C as the origin with the x-axis along CM_0. For a fixed value of α, $0<\alpha<\pi/2$, there are a finite number of vertices M_i', $i=1, 2, \cdots, n$, where n is the greatest integer such that $n\alpha<\pi/2$; and the coordinates of M_i' are

$$x_i = iK\alpha, \quad y_i = K\alpha \sum_{j=1}^{i} \tan j\alpha. \tag{1}$$

Now consider any fixed value of x, say x_0, in the interval $0<x<K\pi/2$; and for any α choose i as the greatest integer such that $iK\alpha \leqq x_0$. Then we have $0 \leqq x_0 - x_i < K\alpha$. Also, from the theory of definite integrals we have

$$\lim_{\alpha \to 0} K\alpha \sum_{j=1}^{i} \tan j\alpha = K \int_0^{x_0/K} \tan t\,dt = K \log \sec (x_0/K). \tag{2}$$

Hence if (x_0, y_0), $0<x_0<K\pi/2$, is a point on the curve

$$y = K \log \sec (x/K), \tag{3}$$

and R is any small region surrounding this point, there will always be within R, for a sufficiently small value of α, a vertex of the set M_i' corresponding to that value of α. It follows that within such a region R there will be infinitely many of the vertices M_i' corresponding to values of α less than some sufficiently small value. The limit curve (3) is such that for very small values of x its form is approximately that of the parabola $2Ky=x^2$.

This curve has a number of interesting properties. It is the curve in which an inextensible cord hangs when its ends are fixed and the area of its cross section at each point is proportional to the tension at that point. For this reason the curve has been called the catenary of uniform strength. If T is the tension at a point where the cross section has the area a, then $K=T/ag\omega$, where ω is the density of the cord. The curve is symmetrical with respect to the vertical line through its lowest point C, the y-axis; and for very small values of α the broken line of the problem $C M_1 M_2' M_3' \cdots$ is an approximation to the part of this curve to the right of the vertical.

Another interpretation is as follows: Suppose that rays of light descend parallel to the y-axis and are reflected from infinitesmal plane mirrors attached one to each segment, say at its right hand end, and with its plane perpendicular to the xy-plane so that its trace on that plane is part of the segment. If $M_k' M'_{k+1}$ is a segment, and the reflected rays from M_k' and M'_{k+1} meet in Q_k', then it is easily seen from a figure that

$$M_k'Q_k' = \delta_k' = \frac{p}{\alpha} \frac{\alpha}{\sin 2\alpha},$$

$$\lim_{\alpha \to 0} \delta_k' = K/2 = \delta.$$

The locus of the limit of Q_k' is the caustic by reflection of the catenary of uniform strength. Hence the latter curve cuts off a constant length of $K/2$ from the tangents to its caustic.

The curve is also a solution of a very simple problem in the calculus of variations. Consider again rays descending parallel to the y-axis, and curves tangent to the x-axis at the origin and cutting the fixed vertical ordinate at x_2 with a given inclination τ_2, $\pi/2>\tau_2>0$. It will also be assumed that at each point of the curve x increases with the inclination τ at that point. Consider also the caustic of each such curve and the area A bounded by the curve, its caustic, and the two reflected rays, one at the origin and the other at the intercept on the fixed vertical at x_2. Then the minimum area is given by the catenary of uniform strength. The proof is elementary. It is easily shown that for any curve $\delta=\frac{1}{2}dx/d\tau$, where δ is again the length of the reflected ray from the curve to its caustic at a point where the inclination of the curve is τ. Hence

$$A = \int_0^{\tau_2} \delta^2 d\tau, \quad x_2 = 2\int_0^{\tau_2} \delta d\tau.$$

We may also write

$$A = \int_0^{\tau_2} \left(\delta - \frac{x_2}{2\tau_2}\right)^2 d\tau + \frac{x_2}{\tau_2}\int_0^{\tau_2} \delta d\tau - \frac{x_2^2}{4\tau_2},$$

$$= \int_0^{\tau_2} \left(\delta - \frac{x_2}{2\tau_2}\right)^2 d\tau + \frac{x_2^2}{4\tau_2}.$$

Therefore the minimum value of A is $x_2^2/4\tau_2$, and this results by taking $\delta =$ a constant $= x_2/2\tau_2$. The equations of the curve easily follow by integration

$$x = 2\delta\tau, \; y = 2\delta \log \sec \tau, \; \delta = x_2/2\tau_2.$$

Solved also by F. Underwood.

3567. [1932, 490] *Proposed by Leverett Davis, University of Washington.*

Find the line through a given point that shall form with two given lines a triangle of minimum area. Characterize the solution geometrically.

Solution by J. W. Querry, University of Iowa.

Denote the angle between the two given lines, l_1 and l_2, by α. Using oblique coordinates with angle of inclination α, take l_1 and l_2 as the x and y axes respectively. Let the given point, P, have coordinates (a, b). Then the third side l of the triangle, Δ, is $y-b=m(x-a)$ where m is to be determined in order to make Δ a minimum. Let l intersect l_1 at A and l_2 at B. Then the coordinates of A and B are $[m^{-1}(am-b), 0]$ and $(0, b-am)$, respectively. Hence the area of the triangle is

$$\Delta = -\frac{\sin\alpha(am-b)^2}{2m},$$

and

$$\frac{d\Delta}{dm} = \frac{\sin\alpha}{2m^2}(b-am)(am+b)$$

Thus the critical values are $m=b/a$ and $m=-b/a$. A trivial solution results when $m=b/a$, while $m=-b/a$ gives the required minimum when one considers only the absolute value of Δ.

The required line l is, therefore, $x/a+y/b=2$.

The geometric construction of l is now obvious. It is to be observed that $BP=PA$ or in other words the required line segment between the two given lines must be bisected by the given point for the triangle to be a minimum.

Solved also by W. B. Campbell, A. G. Clark, J. T. McCarthy, Roy MacKay, M. Markowitz, A. Pelletier, J. Rosenbaum, F. Underwood, and R. M. Winger.

A Note by Otto Dunkel. This problem can be easily solved by elementary geometry. As is seen from the above solution there are triangles having a negative area analytically, and the analysis gives a maximum of zero for such triangles. It is obvious from a figure that in such triangles the fixed point does not lie within the segment of the side which passes through that point. Setting aside this case which is of no interest, we may state the problem as follows:

Given that P is a fixed point lying within the fixed angle $MON < 180°$, and that MN is any straight line through P cutting the sides OM and ON of the angle in M and N, it is required to determine the position of MN so that the area of the triangle OMN is a minimum.

Through P draw a line parallel to MO cutting ON in Q. Then for an actual triangle N must lie on OQ produced. In order to compare the lengths of the segments PN and PM lay off on OM the length $OA = 2QP$, and draw AP cutting ON in B. Then $PB = PA$. Suppose first that N lies on QB produced. The exterior angle at B of the triangle OAB is greater than the interior angle at A. Hence if we rotate the triangle PAM through 180° about P, A will fall upon B and PM will fall along the line of PN. Since M is within OA, AM will fall within the interior of triangle PBN. Thus $PN > PM$, and $PBN > PAM$ where areas are compared in the last inequality. If N lies within QB, M lies on OA produced. Also angle $PAM >$ angle PBN. Hence after rotation of PAM the point M falls on PN produced. Thus $PM > PN$, and $PAM > PBN$.

Therefore in the first case I, and the second case II

I. $OMN - OAB = PBN - PAM > 0$,
II. $OMN - OAB = PAM - PBN > 0$,

where again areas are designated. This shows that OAB has the smallest possible area.

3569. [1932, 490] *Proposed by Nathan Altshiller-Court, University of Oklahoma.*

Through a given line s to draw a plane cutting the faces of a given dihedral angle along the lines a, b, and the bisecting planes of this dihedral angle along the lines c, d, so that the lines c, d shall be the bisectors of the angles formed by a, b. The problem has, in general, two solutions. What are the special cases?

Solution by S. Vatriquant, Athénée Royale d'Ixelles, Brussels, Belgium.

Let α, β be the faces of the given dihedral angle, γ, δ, the two bisecting planes, and A, B, C, D, the points of intersection of these planes with the given line s. If the required plane cuts in X the edge of the dihedral angle, angle CXD is 90°, and a locus of X will be the sphere described on CD as diameter.

We have, in general, two solutions, namely, if the distance of the mid-point of CD to the edge of the dihedral angle is less than $CD/2$; one solution, if the same distance $= CD/2$, and no solution if that distance is more than $CD/2$.

But this last case cannot occur, for the plane through CD and the shortest distance cuts the right dihedral angle $\gamma\delta$ along an angle $>90°$, and the median of the obtained triangle is less than or equal to $CD/2$. The second case occurs when s and the edge of the dihedral angle are orthogonal (1 sol.).

No solution, if s is parallel to the edge of the dihedral angle.

If s is parallel to a single face of the given dihedral angle, e.g. α, A is at infinity, and B is the mid-point of CD. The construction is the same as above.

If s is parallel to a single bisecting plane, e.g. δ, D is at infinity and we have $XA = XB$. Hence a locus of X is the mediating plane of AB, namely, the perpendicular plane to AB through C; we have in this case generally one solution, but if s is perpendicular to the bisecting plane γ, we have ∞ solutions.

Solved also by W. B. Campbell, J. M. Feld, A. Pelletier, and W. H. Rasche.

3571. [1932, 491] *Proposed by Lester R. Ford, Rice Institute.*

What should one pay in order to receive 1, 8, 27, $\cdots$, n^3, $\cdots$, dollars at the end of 1, 2, 3, $\cdots$, n, $\cdots$, years, the interest rate being six percent?

Solution by Robert E. Moritz, The University of Washington.

It is assumed that what is sought in this problem is the limit of the sum

$$S_n = v + 2^3v^2 + 3^3v^3 + \cdots + n^3v^n,\ v = (1.06)^{-1},$$

as n approaches infinity. The existence of the limit may be inferred from the fact that the corresponding infinite series is convergent, the Cauchy test ratio approaching v in the limit.

The sum of the first n terms is readily obtained by multiplying both sides of the above equality by $(1-v)^4$. We find

$$\begin{aligned}(1-v)^4S_n = {} & v+2^3v^2+3^3v^3+4^3v^4+\cdots+n^3v^n \\ & -4v^2-4\cdot 2^3v^3-4\cdot 3^3v^4-\cdots-4n^3v^{n+1} \\ & +6v^3+6\cdot 2^3v^4+\cdots+6(n-1)^3v^{n+1}+6n^3v^{n+2} \\ & -4v^4-\cdots-4(n-2)^3v^{n+1}-4(n-1)^3v^{n+2}-4n^3v^{n+3} \\ & v^5+\cdots+(n-3)^3v^{n+1}+(n-2)^3v^{n+2}+(n-1)^3v^{n+3}+n^3v^{n+4}.\end{aligned}$$

The coefficient of v^k for $k = 5, 6, 7, \cdots, n$ is equal to

$$k^3 - 4(k-1)^3 + 6(k-2)^3 - 4(k-3)^3 + (k-4)^3$$

which vanishes for all values of k. Collecting the remaining coefficients we have

$$S_n = \frac{v + 4v^2 + v^3}{(1-v)^4} - \frac{(n+1)^3v^{n+1} - (3n^3+6n^2-4)v^{n+2} + (3n^3+3n^2-3n+1)v^{n+3} - n^3v^{n+4}}{(1-v)^4}.$$

L'Hospital's Theorem shows that as n approaches infinity the second part of the expression on the right approaches 0 as a limit. Therefore

$$S = \lim S_n = \frac{v + 4v^2 + v^3}{(1 - v)^4}.$$

Since $v = (1.06)^{-1}$, we readily find $S = 520{,}479.63$, the number of dollars of the present value sought.

Solved also by W. B. Campbell, A. G. Clark, J. D. Leith, F. L. Manning, A. S. Merrill, H. A. Meyer, and F. Underwood.

A Note by Otto Dunkel. The finite series of this problem belongs to a more general class which can be summed by a process which may be of interest since it is quite analogous to integration by parts. Let $\sum$ denote the operation of summing, and let Δ denote the difference operator with the unit difference of unity. Then for any two functions $u(x)$ and $v(x)$ the formula

$$\Delta u(x)v(x) = u(x + 1)\Delta v(x) + v(x)\Delta u(x) \tag{1}$$

is obvious. Hence by summation we have

$$\sum^{n} v(x)\Delta u(x) = u(n + 1)v(n + 1) - \sum^{n} u(x + 1)\Delta v(x), \tag{2}$$

where a constant is to be added on the right depending upon the lower limit for x. This is our formula for summation by parts.

The example in the problem is of the type in which the terms to be summed are of the form $P(x)a^x$, where $P(x)$ is a polynomial in x of degree k and a is a constant not unity. Since $\Delta a^x = (a-1)a^x$, we have from (2)

$$\sum^{n} P(x)a^k = \sum^{n} \frac{P(x)}{a - 1}\Delta a^x = \frac{P(n + 1)}{a - 1}a^{n+1} - \sum^{n} \frac{\Delta P(x)}{a - 1}a^{x+1}. \tag{3}$$

Repeating the process (3) on the second term on the right, and continuing in this way using (3) as a recursion formula, we have

$$\begin{aligned}\sum^{n} P(x)a^x = {} & \frac{P(n + 1)}{a - 1}a^{n+1} - \frac{\Delta P(n + 1)}{(a - 1)^2}a^{n+2} + \frac{\Delta^2 P(n + 1)}{(a - 1)^3}a^{n+3} - \cdots \\ & + (-1)^k \frac{\Delta^k P(n + 1)}{(a - 1)^{k+1}}a^{n+k+1}.\end{aligned} \tag{4}$$

Setting $P(x) = x^3$, we have for the given example

$$\sum_{0}^{n} x^3a^x = \frac{a^{n+1}}{(a-1)}\left[(n + 1)^3 - \frac{a}{a - 1}(3n^2 + 9n + 7) + \frac{6a^2}{(a - 1)^2}(n + 2) - \frac{6a^3}{(a - 1)^3}\right] + C$$

The value of C is found by setting $n = 0$, and we obtain

$$C = \frac{a^3 + 4a^2 + a}{(a - 1)^4}.$$

Finite series of the form

$$\sum^{n} P(x) \sin (ax + b),$$

where a and b are constants and $P(x)$ is again a polynomial of the kth degree, may also be summed by the same method. For we may write

$$\sin (ax + b) = \frac{\Delta \sin [ax + b - \frac{1}{2}(a + \pi)]}{2 \sin (a/2)},$$

and the summation proceeds as before.

3572. [1932, 549] *Proposed by Nathan Altshiller-Court, University of Oklahoma.*

The sum of the three bimedians of a tetrahedron (i.e., the lines joining the mid-points of the pairs of opposite edges) is less than one half and greater than one fourth of the sum of the edges of the tetrahedron.

Solution by J. Rosenbaum, Milford, Conn.

Let the sides of a face be a, b, c, and the opposite edges, x, y, z. Also let the mid-points of a, b, c be P_1, P_2, P_3, and the mid-points of x, y, z; Q_1, Q_2, Q_3. From the triangle $P_2Q_2Q_3$ we have $P_2Q_2 < Q_2Q_3 + Q_3P_2$; and there are two other inequalities which follow in the same manner. These may be written

$$P_2Q_2 < (a + x)/2,\ P_3Q_3 < (b + y)/2,\ P_1Q_1 < (c + z)/2.$$

The addition of these three inequalities yields the first part of the required inequality.

The second part is proved by making use of the theorem that the bimedians of a tetrahedron are concurrent at a point O which bisects each bimedian. Thus from the triangle OP_2P_3 we have $OP_2 + OP_3 > P_2P_3$, or $(P_2Q_2 + P_3Q_3)/2 > a/2$. There are two other inequalities which are obtained in a similar manner: and these may be written

$$P_2Q_2 + P_3Q_3 > a,\ P_3Q_3 + P_1Q_1 > b,\ P_1Q_1 + P_2Q_2 > c.$$

Adding these three inequalities we have

$$P_1Q_1 + P_2Q_2 + P_3Q_3 > (a + b + c)/2.$$

In the same manner by using triangles such as OP_1Q_3, we find

$$P_1Q_1 + P_2Q_2 + P_3Q_3 > (x + y + z)/2.$$

The addition of these last two inequalities gives the second required inequality.

Solved also by A. D. Bradley, Rufus Crane, L. S. Johnston, M. Markowitz, J. B. Meyer, H. D. Ruderman, C. A. Rupp, and F. Underwood.

3573. [1932, 549] *Proposed by Samuel I. Jones, Nashville, Tenn.*

A hawk, eagle, and sparrow are in the air. The eagle is 50 feet above the sparrow and the hawk is 100 feet below the sparrow. The sparrow flies straight forward in a horizontal line. Both hawk and eagle fly directly towards the sparrow. The hawk flies twice as fast as the sparrow. The hawk and eagle reach the sparrow at the same time. How far does each fly and at what rate does the eagle fly?

Solution by Eugene M. Berry, Lynchburg College.

First let us consider the curve of pursuit followed by either the hawk or the eagle. Let the sparrow start at the origin with a speed of unity and let $1/k$ be the speed and s the distance traveled by the pursuer. At the time $t=0$, the position of the pursuer is $(0, b)$; at any time, t, the position of the sparrow is $(t, 0)$, while that of the pursuer is (x, y). Then $t=ks$.

Since the hawk or eagle flies directly toward the sparrow we have $dx/dy = (x-t)/y = (x-ks)/y$. Differentiate this with respect to y and put q for dx/dy, and, using the fact that $ds/dy = -\sqrt{1+q^2}$, we get

$$y(dq/dy) = k\sqrt{1+q^2}.$$

The solution of this is

$$q + \sqrt{1+q^2} = Ay^k, \tag{1}$$

A is positive although q is negative. From the initial conditions, $q=0$ when $y=b$, we find $A=b^{-k}$. Using this and solving (1) for q we get

$$q = dx/dy = (b^{-k}y^k - b^k y^{-k})/2.$$

Integrating, we get

$$x = \tfrac{1}{2}[b^{-k}y^{1+k}/(1+k) - b^k y^{1-k}/(1-k)] + c. \tag{2}$$

Since $y=b$ when $x=0$, $c=bk/(1-k^2)$. The conditions at the end are $x=a, y=0$. Using these values and the value for c in equation (2) we get

$$a = bk/(1-k^2). \tag{3}$$

It is evident that

$$s = a/k. \tag{4}$$

Solving equation (3) for k we get

$$k = (-b + \sqrt{b^2+4a^2})/2a. \tag{5}$$

Only the positive sign is used before the radical since k is positive and numerically less than 1, since we assume the sparrow is to be caught.

For the case of the hawk $k=1/2$, $b=100$. Then from equation (3) we get $a=66\ 2/3$ and equation (4) gives $s=133\ 1/3$.

For the case of the eagle, putting $a=66\ 2/3$ and $b=50$ in equation (5) gives $k=.693$ and $1/k=1.443$; then equation (4) gives $s=96.2$.

Thus, the sparrow travels 66 2/3 ft., the hawk 133 1/3 ft. and the eagle 96.2 ft. The speed of the eagle is 1.443 times that of the sparrow.

Solved also by J. A. Calderhead, Paul Capron, J. P. Howe, J. D. Leith, M. Markowitz, W. H. Turney, and F. Underwood.

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending items to Professor J. H. Weaver, Ohio State University, Columbus, Ohio.

THE ANNUAL MEETING OF THE MATHEMATICAL ASSOCIATION

The eighteenth annual meeting of the Mathematical Association of America will be held in Cambridge, Massachusetts, December 27–30, 1933, in conjunction with the meetings of the American Association for the Advancement of Science and the American Mathematical Society. The sessions of the Society will be held on Wednesday and Thursday, those of the Mathematical Association on Friday and Saturday, and there will be a joint session with Section A of the American Association and the Society on Friday morning.

Arrangements have been made for members to be accommodated at the Commander Hotel and the Continental Hotel, Cambridge. Headquarters for mathematicians and their families will be at the Commander Hotel. The scientific sessions of the Society and the Association will be held in Longfellow Hall, Radcliffe College, Cambridge.

A joint dinner of the mathematical organizations will be held on Thursday evening at the Walker Memorial Building, Massachusetts Institute of Technology, for which the charge will be $1.50 per person. Before the dinner, there will be an exhibition of machines, charts, and the electric integraph.

Through the kindness of Professor and Mrs. J. L. Coolidge, mathematicians and their friends will be entertained at tea on Wednesday afternoon at Lowell House, Harvard University.

Special rates will be in force at the Commander Hotel and the Continental Hotel. Every room is equipped with a private bath. The charges are: single rooms, $3.00 per day; double rooms, $2.25 per day, per person. Meals are available in the hotels at reasonable prices. Those desiring accommodations at more modest prices will be able to obtain them nearby.

The usual railway rates (to Boston) of a fare and a half for return will doubtless be available.

The "Annali della R. Scuola Normale Superiore di Pisa-mathematical and physical sciences," announces a competition on the following themes:

I. Prepare a rapid digest of the theory of partial differential equations of elliptic and parabolic types.

II. Prepare a rapid digest of the theory of analytic functions of several complex variables, and of particular classes of especially remarkable functions of that kind.

To the best work on either of these two themes, which is received by the board of editors of the "Annali" (Direzione degli Annali, Pisa, Scuola Normale Superiore) not later than June 15, 1934 there will be given a prize of two thousand Lire (2000 L), of which one thousand were given expressly for this purpose by a former student of the Scuola who wishes to remain incognito. For this gift the board of directors expresses its profound thanks.

Only works not previously published will be accepted. They must be typewritten and three duplicate copies submitted, and they must be written in Italian, French, English or German, and not exceed thirty printed pages of the "Annali."

The memoir receiving the prize must be translated into Italian under the auspices of its author. This translation will be published in the "Annali."

The competition will be conducted by a commission, which will be appointed at the proper time by the board of directors of the "Annali."

Professors Lipot Fejér and Tullio Levi-Civita arrived in this country several weeks before the Society's meeting at Chicago in June, at which they both lectured. Professor Fejér lectured during April and May at Brown University, Harvard, Columbia, Princeton, Ohio State, the University of Pennsylvania, Cornell, and the University of Cincinnati, and before a sectional meeting of the Mathematical Association of America at the University of Virginia. His topics were "On new properties of the arithmetical means of the partial sums of Fourier Series" and "On the characterization of some remarkable systems of points of interpolation by means of conjugate points of the Cotes' numbers; and of certain extremal properties." His lectures at the Chicago meeting were "On the infinite sequences arising in the theories of harmonic analysis, of interpolation, and of mechanical quadratures." Professor Levi-Civita lectured at Princeton Harvard and Brown during the last week in May and the first week in June, on "Secular effects of the tides on the motion of a planetary system" and "On adiabatic invariants." His lectures at Chicago were "On some mathematical aspects of the new mechanics" and "Nets on a surface and extension of trigonometry."

Dr. Felix Bernstein, director of the Institute of Mathematical Statistics of the University of Göttingen, spoke before the Sigma Xi chapter at the University of Cincinnati and the mathematics club of the Ohio State University on April 10 and 11 on the subject "A solution of mathematical problems in physics and engineering by new mechanical means."

Professor Albert Einstein has resigned his membership in the Prussian

Academy of Sciences. He has accepted professorial appointments at the Universities of Madrid and Paris in addition to his professorship at the Institute for Advanced Study at Princeton.

Dr. Irving Langmuir of the General Electric Company was made an honorary member of the School of Engineering Alumni Association at Columbia University on April 26.

Dr. Henry Norris Russell, research professor in astronomy and director of the observatory at Princeton, president this year of the American Association for the Advancement of Science, delivered the Halle lecture at the University of Oxford, June 1.

At the April meeting of the National Academy of Sciences in Washington Professors G. C. Evans of Rice Institute and J. F. Ritt of Columbia University were elected members of the academy. At this meeting the following mathematical papers were read: *Geometry of the Laplace equation*, by Professor Edward Kasner; *The geometry of spinors*, by Oswald Veblen.

Professors Jesse Douglas and M. H. Stone have been elected fellows of the American Academy of Arts and Sciences.

Professor E. V. Huntington, of Harvard University, has been elected a member of the American Philosophical Society.

Professor W. H. Roever, of Washington University, has received a grant from the Rockefeller Science Research Fund for the publication of some of his work in descriptive geometry. Part of this work was done for the Committee on Standards for Graphical Presentation, on which Professor Roever served as representative of the American Mathematical Society.

The council of the Royal Society of London has recommended Professor J. H. M. Wedderburn of Princeton University for membership in that society.

Professor R. L. Wilder of the University of Michigan has been granted leave of absence for the next academic year in order to continue work at the Institute for Advanced Study at Princeton.

Professor T. C. Esty of the department of mathematics at Amherst College has been made vice-president of the college.

Associate Professor W. C. Graustein of Harvard University has been promoted to a professorship of mathematics.

Associate Professor Einar Hille of Princeton University has been appointed professor of mathematics at Yale University.

Assistant Professor M. H. Stone of Yale University has been appointed associate professor of mathematics at Harvard University.

Dr. H. F. Bohnenblust has been appointed assistant professor of mathematics at Princeton.

Dr. J. J. Gergen has been appointed to an assistant professorship at the University of Rochester.

The following appointments to instructorships in mathematics are announced:

Harvard University—D. P. Adams, J. H. Curtiss, T. L. Downs, Jr., Z. I. Mosesson, Arthur Sard, F. H. Steen, D. H. Ballou, Walter Leighton, Jr., G. B. VanSchaack, Hassler Whitney. In addition to these, Wladimir Seidel and A. E. Currier have been appointed Benjamin Pierce Instructors in Mathematics and Tutors in the Division of Mathematics.

Princeton University—E. J. McShane and A. W. Tucker.
University of Rochester—G. B. Price.

Dr. M. F. Deuring and Mr. Saunders MacLane have been awarded Sterling fellowships in mathematics at Yale.

Professor L. M. Defoe, professor emeritus of mechanics at the University of Missouri, died April 3, 1933, at the age of 72.

Mr. R. A. E. C. Paley, international research fellow at Massachusetts Institute of Technology, was killed near Banff, Alberta, by an avalanche April 7, 1933. Mr. Paley was one of the most promising of the young English mathematicians. He had done considerable work with Professor Littlewood and was continuing his work with Professor Wiener at the Institute.

Dr. J. G. Porter, Professor of astronomy at the University of Cincinnati and director of the observatory from 1884 to 1931, died April 15, 1933.

Professor F. B. Williams of Clark University died March 7, 1933. He was a charter member of the Mathematical Association.

A NOTE

Mr. S. A. Corey has called my attention to the fact that, in my *Reading List in the Elementary Theory of Equations*, published in this MONTHLY, vol. 40, 1933, pp. 77–84, I omitted a paper of his. The title is *A Method of Solving Numerical Equations*, and it was published in this MONTHLY, vol. 21, 1914, pp. 290–92. It should be listed in part V of my classification. Mr. Corey has a more recent paper of the same general type, and bearing the same title, in the MONTHLY for March 1933, pp. 163–64.

RAYMOND GARVER

The Chauvenet Prize

In the year 1925, the Mathematical Association of America established a prize of one hundred dollars for the best expository paper published in English during successive periods of five years by a member of the Association.

The purpose of the prize is to stimulate expository contributions in mathematical journals. The award does not apply to books, although the Carus Monographs are expository in character and on this score might be included. They carry their own reward in the form of a cash honorarium to each author.

It is believed that clear expositions of mathematical subjects are greatly needed in this country and that the Chauvenet Prize will tend to stimulate such production.

The prize will be awarded hereafter every three years. The last award was in December, 1932, to Professor G. H. Hardy. The next award will be in December, 1935, for the period 1931-1934.

Note that the prize is to be awarded only to a member of the Association—one more of the many good reasons for membership.

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Individual and institutional members may procure copies at $20.00 per set through Secretary Cairns at Oberlin, Ohio. All others must order through the Open Court Publishing Company, 339 E. Chicago Avenue, Chicago, Ill., at $25.00 per set.

It is a magnificent work and should be in every college library. The edition is absolutely limited. One-half of the sets are already sold, and no more will be available when this edition is exhausted.

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DIRECTORY

EDITORIAL CORRESPONDENCE should be addressed to the EDITOR-IN-CHIEF, W. B. CARVER, White Hall, Cornell University, Ithaca, N.Y.

BOOKS FOR REVIEW should be addressed to R. A. JOHNSON, Brooklyn College, 66 Court Street, Brooklyn, N.Y.

BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS, 33 Peters Hall, Oberlin, Ohio.

CHANGE OF ADDRESS: Members should send notice of any change of address to the SECRETARY-TREASURER, W. D. CAIRNS, Oberlin, Ohio, before the 10th of each month.

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Seventeenth Summer Meeting of the Association, Chicago, Ill., June 20-22, 1933.

The following is a list of the sections of the Association, with dates of those section meetings which have been scheduled for 1933 and reported to the Secretary.

ILLINOIS, merges with the Chicago meeting.
INDIANA, Bloomington, May 5-6.
IOWA, Cedar Rapids, Apr. 21-22.
KANSAS, Topeka, Feb. 11.
KENTUCKY, May.
LOUISIANA-MISSISSIPPI, Ruston, La., Mar. 3-4.
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Charlottesville, Va., May 13; Washington, D.C., Dec. 2.
MICHIGAN, Ann Arbor, Mar. 18.
MINNESOTA.
MISSOURI.
NEBRASKA, Lincoln, Apr. 28.
OHIO, Columbus, Apr. 6.
PHILADELPHIA, Philadelphia, Dec. 2.
ROCKY MOUNTAIN, Fort Collins, Colo., Apr. 14-15.
SOUTHEASTERN, Athens, Ga., March.
SOUTHERN CALIFORNIA, Claremont, Mar. 4.
TEXAS, Dallas, Feb. 11.
WISCONSIN, Beloit, Apr. 8.

AFFILIATED ORGANIZATIONS: THE NEW ENGLAND ASSOCIATION OF TEACHERS OF MATHEMATICS. THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS.

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THE SEVENTEENTH SUMMER MEETING OF THE MATHEMATICAL ASSOCIATION

The seventeenth summer meeting of the Mathematical Association of America was held, by invitation, at A Century of Progress Exposition, the University of Chicago and Northwestern University, during the week of June 19, 1933, in affiliation with the American Association for the Advancement of Science and the American Mathematical Society. Nearly five hundred persons were present at the meetings, including the following one hundred ninety-three members of the Association:

H. M. ACKLEY, State Teachers College, Kalamazoo, Mich.
C. R. ADAMS Brown University
R. P. AGNEW, Cornell University
R. B. ALLEN, Kenyon College
NOLA L. ANDERSON, Sophie Newcomb College

R. W. BABCOCK, Kansas State Agricultural College
FRANCES E. BAKER, Iowa City, Iowa
G. A. BAKER, Shurtleff College
R. W. BARNARD, University of Chicago
I. A. BARNETT, University of Cincinnati
WALTER BARTKY, University of Chicago
W. D. BATEN, University of Michigan
ETHELWYNN R. BECKWITH, Milwaukee-Downer College
THEODORE BENNETT, University of Wisconsin
B. A. BERNSTEIN, University of California
H. R. BEVERIDGE, Monmouth College
G. D. BIRKHOFF, Harvard University
H. F. BLICHFELDT, Stanford University
G. A. BLISS, University of Chicago
JULIA W. BOWER, University of Chicago
C. F. BOWLES, South Dakota State School of Mines
FANNIE W. BOYCE, Wheaton College
W. E. BROOKE, University of Minnesota
C. T. BUMER, Kenyon College
O. Z. BURDICK, Chicago, Illinois
R. S. BURINGTON, Case School of Applied Science
W. H. BUSSEY, University of Minnesota

W. D. CAIRNS, Oberlin College
HELEN CALKINS, Pennsylvania College for Women
G. A. CAMPBELL, American Tel. and Tel. Co.
J. W. CAMPBELL, University of Alberta
R. D. CARMICHAEL, University of Illinois
J. E. CASE, St. Louis University
W. F. CHENEY, JR., Connecticut Agricultural College
R. V. CHURCHILL, University of Michigan
G. R. CLEMENTS, U. S. Naval Academy
A. B. COBLE, University of Illinois
L. M. COFFIN, Coe College
L. W. COHEN, University of Kentucky
J. B. COLEMAN, University of South Carolina
JULIA T. COLPITTS, Iowa State College
A. R. CRATHORNE, University of Illinois
D. R. CURTISS, Northwestern University
E. H. CUTLER, Lehigh University
L. H. CUTTING, High School, Kansas City, Mo.

W. M. DAVIS, Armour Institute of Technology
L. E. DICKSON, University of Chicago
L. L. DINES, University of Saskatchewan
B. F. DOSTAL, University of Florida
J. E. DOTTERER, Manchester College
ARNOLD DRESDEN, Swarthmore College
W. L. DUREN, JR., Tulane University

W. E. EDINGTON, DePauw University
MARGARET C. EIDE, State Teachers College, River Falls, Wis.
E. B. ESCOTT, Oak Park, Illinois
H. P. EVANS, University of Wisconsin
H. S. EVERETT, University of Chicago

F. J. FEINLER, St. Peter's Church, Loudonville, Ohio
B. F. FINKEL, Drury College
C. H. FISCHER, University of Minnesota
A. H. FOX, Schenectady, N. Y.
T. C. Fry, Bell Telephone Laboratories

M. G. GABA, University of Nebraska
W. H. GARRETT, Baker University
B. P. GILL, College of the City of New York